On the equations of stationary motion of perfectly plastic fluids

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Abstract

The stationary motion of an incompressible fluid is governed by the system of PDEs

$$\nabla \cdot \mathbf{u} = 0, \qquad -\nabla \cdot S + \nabla p = \mathbf{f} \tag{1}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ velocity, $S = \{S_{ij}\}$ deviatoric stress, p pressure, $\mathbf{f}$ external force. We consider the following constitutive law (R. von Mises (1913)):

$$D = 0 \Longrightarrow |S| \leq g, \quad D \neq 0 \Longrightarrow S = \frac{g}{|D|} D \tag{2}$$

($D = D(\mathbf{u}) = \{D_{ij}(\mathbf{u})\}$, $D_{ij}(\mathbf{u}) = \dfrac{1}{2}\,(\partial_j u_i + \partial_i u_j)$ rate of strain, $g = \text{const} > 0$ yield value). The relations (2) model perfect plastic behavior of an incompressible fluid („von Mises solid").

The weak formulation of (1), (2) in a bounded domain $\Omega \subset \mathbb{R}^n$ under Dirichlet boundary conditions on $\mathbf{u}$ leads to the problem

$$\text{minimize} \qquad \mathcal{F}(\mathbf{u}) : = \ g \int_\Omega |D(\mathbf{u})| - \int_\Omega \mathbf{f} \cdot \mathbf{u}.$$

We solve this problem in the space $\mathrm{BD}(\Omega)$ under the assumption that $\mathbf{f}$ satisfies a safe load condition. Moreover, we prove the existence of an admissible stress field S, and present a generalization of (2) in terms of geometric measure theory (joint work with M. Bildhauer).