N_0 completions on partial matrices ⋆

C. Jordán , C. Mendes Araújo , Juan R. Torregrosa *

Instituto de Matemática Multidisciplinar / Centro de Matemática
Universidad Politécnica de Valencia / Universidade do Minho
Camino de Vera, s/n, 46022 Valencia / Campus de Gualtar, 4710 Braga
Spain / Portugal

Abstract

An $n \times n$ matrix is called an N_0-matrix if all its principal minors are nonpositive. In this paper, we are interested in N_0-matrix completion problems, that is, when a partial N_0-matrix has an N_0-matrix completion. In general, a combinatorially or non-combinatorially symmetric partial N_0-matrix does not have an N_0-matrix completion. Here, we prove that a combinatorially symmetric partial N_0-matrix, with no null main diagonal entries, has an N_0-matrix completion if the graph of its specified entries is a 1-chordal graph or a cycle. We also analyze the mentioned problem when the partial matrix has some null main diagonal entries.

Key words: Partial matrix, completion, N_0-matrix, chordal graph, cycle.

1 Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not. We make the assumption throughout that all diagonal entries are prescribed. A *completion* of a partial matrix is the matrix resulting from a particular choice of values for the unspecified entries. The completion obtained by replacing all the unspecified entries by zeros is called the *zero completion* and denoted A_0. A *completion problem* asks if we can obtain a completion of a partial matrix with some prescribed properties. An $n \times n$ partial matrix

⋆ This research was supported by the Portuguese Foundation for Science and Technology (FCT) through the research program POCTI and by Spanish DGI grant number MTM2007-64477

* Correspondig author
Email addresses: `cjordan@mat.upv.es` (C. Jordán), `clmendes@math.uminho.pt` (C. Mendes Araújo), `jrtorre@mat.upv.es` (Juan R. Torregrosa).

$A = (a_{ij})$ it said to be *combinatorially symmetric* when a_{ij} is specified if and only if a_{ji} is, and is said to be *weakly sign-symmetric* if $a_{ij}a_{ji} \geq 0$, for all $i, j \in \{1, 2, \ldots, n\}$ such that both (i, j), (j, i) entries are specified.

A natural way to described an $n \times n$ partial matrix $A = (a_{ij})$ is via a graph $G_A = (V, E)$, where the set of vertices V is $\{1, 2, \ldots, n\}$, and the edge or arc $\{i, j\}, (i \neq j)$ is in set E if and only if position (i, j) is specified; as all main diagonal entries are specified, we omit loops. In general, a directed graph is associated with a non-combinatorially symmetric partial matrix and, when the partial matrix is combinatorially symmetric, an undirected graph is used.

A *path* is a sequence of edges (arcs) $\{i_1, i_2\}$, $\{i_2, i_3\}$, $\ldots$, $\{i_{k-1}, i_k\}$ in which the vertices are distinct. A *cycle* is a path with the first vertex equal to the last vertex. An undirected graph is *chordal* if it has no induced cycles of length 4 or more [1].

The submatrix of a matrix A, of size $n \times n$, lying in rows α and columns β, $\alpha, \beta \subseteq N = \{1, 2, \ldots, n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$.

An $n \times n$ real matrix $A = (a_{ij})$ is called N_0-*matrix* (N-matrix) if all its principal minors are nonpositive (negative). These classes of matrices arise in multivariate analysis [6], in linear complementary problems [4,5] and in the theory of global univalence of functions [2].

In the following proposition we give some properties that are very useful in the study of N_0-matrices.

Proposition 1 *Let $A = (a_{ij})$ be an $n \times n$ N_0-matrix. Then*

(1) If P is a permutation matrix, then PAP^T is an N_0-matrix.
(2) If D is a positive diagonal matrix, then DA and AD are N_0-matrices.
(3) If D is a nonsingular diagonal matrix, then DAD^{-1} is an N_0-matrix.
(4) If $a_{ii} \neq 0$, $i = 1, 2, \ldots, n$, then $a_{ij} \neq 0$, $\forall i, j \in \{1, 2, \ldots, n\}$.
(5) A is weakly sign-symmetric
(6) $\forall \alpha \subset \{1, 2, \ldots, n\}$, principal submatrix $A[\alpha]$ is an N_0-matrix.

From the above properties, it is easy to prove that any $n \times n$ N_0-matrix, with no null diagonal entries, is diagonally similar to an N_0-matrix in the set:

$$S_n = \{A = (a_{ij}) : \ a_{ij} \neq 0 \text{ and } sign(a_{ij}) = (-1)^{i+j+1}, \ \forall i, j\}$$

On the other hand, the last property of the previous proposition allows us to give the following definition.

Definition 1 *A partial matrix is said to be a partial N_0-matrix if every com-*

pletely specified principal submatrix is an N_0-matrix.

In [3] the authors study the N-matrix completion problem, and they close the problem for some types of partial matrices. Our interest here is in the N_0-matrix completion problem, that is does a partial N_0-matrix have an N_0-matrix completion? The study of this problem is different from the previous one since some minors or some entries of the partial matrix can be zero.

In Section 2 we analyze this problem for combinatorially and non-combinatorially symmetric partial N_0-matrices. The problem has, in general, a negative answer. In Section 3 we show that the 1-chordal graphs guarantee the existence of an N_0-matrix completion for a partial N_0-matrix and in Section 4 we complete the study of the mentioned completion problem for a partial N_0-matrix whose associated graph is a cycle. In both cases we consider partial N_0-matrix with no null main diagonal entries. Finally, in Section 5 we analyze the problems that appear when some main diagonal entries are zero.

2 Preliminary results

The following example shows that the N_0-matrix completion problem has, in general, a negative answer for combinatorially and non-combinatorially symmetric partial N_0-matrices.

Example 1 The partial matrices

$$A = \begin{bmatrix} -1 & 1 & x \\ 1 & 0 & 0 \\ y & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & x & 3 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

are combinatorially and non-combinatorially symmetric partial N_0-matrices, respectively. However, A has no N_0-matrix completion since $\det A = 1$, $\forall x, y$. On the other hand, $\det B[\{1,2\}] \leq 0$ if and only if $x \geq 1/2$, but $\det B > 0$, $\forall x \geq 1/2$. Therefore, B has no N_0-matrix completion.

We produce a partial N_0-matrix of size $n \times n$, $n \geq 4$, which has no N_0-matrix completion, by embedding the above matrix A (analogously B) as a principal submatrix and putting -1's on the main diagonal and unspecified entries on the remaining positions, that is

$$M = \begin{bmatrix} A & X \\ Y & \bar{I} \end{bmatrix},$$

where $\bar{I}$ is a partial matrix with all entries unspecified except the entries of the main diagonal that are equal to -1, and X and Y are completely unspecified submatrices.

Keeping Proposition 1 in mind, when we suppose that all diagonal entries are non-zero, it would not make sense to study the existence of N_0-matrix completion of non-weakly sign-symmetric partial N_0-matrices or of partial N_0-matrices which do not satisfy, up to permutation and diagonal similarity, that if (i,j) entry is specified, then $sign(a_{ij}) = (-1)^{i+j+1}$. Therefore, in this context (partial matrices with non-zero diagonal entries) we assume that the partial N_0-matrix belongs to the set PS_n of partial matrices such that for all (i,j) specified entry $sign(a_{ij}) = (-1)^{i+j+1}$.

When restricting our study to partial N_0-matrices that belong to PS_n, we are implicitly analyzing the completion problem for partial N_0-matrices that are permutation or diagonally similar to a partial N_0-matrix that belongs to PS_n.

Proposition 2 *Let A be a partial N_0-matrix of size 3×3 with no null main diagonal entries. Then, there exists an N_0-matrix completion A_c of A.*

Proof: Since the class of N_0-matrices is invariant under left and right positive diagonal multiplication, we may take all diagonal entries of A to be -1. We denote by λ_A the number of unspecified entries of A. If $\lambda_A = 0$, A is not a partial matrix and if $\lambda_A = 6$, the result is trivial.

Let us first consider the case in which A has exactly one unspecified entry. By permutation and diagonal similarities, we can assume that this entry is in position (1,3) and that all upper diagonal entries are equal to 1. Hence, A has the following form

$$A = \begin{bmatrix} -1 & 1 & -x \\ a_{21} & -1 & 1 \\ -a_{31} & a_{32} & -1 \end{bmatrix},$$

with $a_{21}, a_{32} \geq 1$ and $a_{31} > 0$.

If $a_{31} > 1$, it suffices to choose $x = 1$ in order to obtain an N_0-matrix completion of A. If $a_{31} \leq 1$, we consider the completion A_c of A with $x = 1/a_{31}^2$.

The formulation of the problem in case $\lambda_A > 1$ reduces to that of $\lambda_A = 1$. In fact, it is possible to complete some adequate unspecified entries in order to obtain a partial N_0-matrix with a single unspecified entry. $\square$

Unfortunately, we can not extend Proposition 2 for general n, as we see in the following example.

Example 2 Let A be the partial matrix

$$A = \begin{bmatrix} -1 & 1 & -10 & x \\ 2 & -1 & 1 & -100 \\ -0.1 & 10 & -1 & 1 \\ 1 & -10 & 1 & -1 \end{bmatrix}.$$

It is not difficult to verify that A is a partial N_0-matrix and $A \in PS_4$. However, A has no N_0-matrix completion, since $\det A[\{1,2,4\}] = 901 - 19x \leq 0$ and $\det A[\{1,3,4\}] = 0.9x - 9 \leq 0$ if and only if $x \geq 901/19$ and $x \leq 10$, which is impossible.

From this example, we can establish de following result.

Proposition 3 *For every $n \geq 4$, there is an $n \times n$ non-combinatorially symmetric, partial N_0-matrix belonging to PS_n, that has no N_0-matrix completion.*

In Sections 3 and 4 we are going to work with combinatorially symmetric partial N_0-matrices, which belong to PS_n and with no null main diagonal entries.

3 Chordal graphs

We recall some very rich clique structure of chordal graphs (see [1] for further information). A *clique* in an undirected graph G is simply a complete (all possible edges) induced subgraph. We also use clique to refer to a complete graph and use K_p to indicate a clique on p vertices.

If G_1 is the clique K_q and G_2 is any chordal graph containing the clique K_p, $p < q$, then the *clique sum* of G_1 and G_2 along K_p is also chordal. The cliques that are used to build chordal graphs are the maximal cliques of the resulting chordal graph and the cliques along which the summing takes place are the so-called *minimal vertex separators* of the resulting chordal graph. If the maximum number of vertices in a minimal vertex separator is p, then the chordal graph is called *p-chordal.* In this section we are interested in 1-chordal graphs.

Proposition 4 *Let A be an $n \times n$ partial N_0-matrix, with no null main diagonal entries, the graph of whose specified entries is 1-chordal with two maximal cliques, one of them with two vertices. Then, there exists an N_0-matrix completion of A.*

Proof: Since the class of N_0-matrices is invariant under left and right positive diagonal multiplication, we may take all diagonal entries of A to be -1. Moreover, since there are no null main diagonal entries, A is permutationally similar to a partial N_0-matrix in PS_n. Therefore, we may assume, without loss of generality, that $A = (\bar{a}_{ij})$ has the following form:

$$A = \begin{bmatrix} -1 & 1 & x_{13} & \cdots & x_{1n} \\ a_{21} & -1 & 1 & \cdots & (-1)^{n+1}a_{2n} \\ x_{31} & a_{32} & -1 & \cdots & (-1)^{n+2}a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & (-1)^{n+1}a_{n2} & (-1)^{n+2}a_{n3} & \cdots & -1 \end{bmatrix},$$

Consider the completion A_c of A obtained by replacing the unspecified entries in the following way:

$$x_{1j} = -\bar{a}_{2j}, \quad j \in \{3, 4, \ldots, n\}$$
$$x_{i1} = -\bar{a}_{i2}, \quad i \in \{3, 4, \ldots, n\}$$

To prove that A_c is an N_0-matrix, we only need to show that $\det A_c[\{1\} \cup \alpha] \leq 0$, for all $\alpha \subseteq \{2, \ldots, n\}$. In order to do so, let $\alpha \subseteq \{2, \ldots, n\}$. If $2 \in \alpha$, $\det A_c[\{1\} \cup \alpha] = (a_{21} - 1) \det A[\alpha] \leq 0$. If $2 \notin \alpha$, $A_c[\{1\} \cup \alpha]$ is obtained from $A_c[\{2\} \cup \alpha]$ by multiplying the first row and the first column by -1. Then, $\det A_c[\{1\} \cup \alpha] = \det A_c[\{2\} \cup \alpha] \leq 0$ □

Despite the preceding result being a particular case of the next proposition, its proof will be very useful in the resolution of completion problems for certain partial N_0-matrices with special types of associated graphs.

Proposition 5 *Let A be an $n \times n$ partial N_0-matrix, with no null main diagonal entries, the graph of whose specified entries is 1-chordal with two maximal cliques. Then, there exists an N_0-matrix completion of A.*

Proof: Taking into account proposition 1, we may assume, without loss of generality, that A has the following form

$$A = \begin{bmatrix} A_{11} & a_{12} & X \\ a_{21}^T & -1 & a_{23}^T \\ Y & a_{32} & A_{33} \end{bmatrix},$$

where X and Y are completely unspecified matrices and the remaining entries of A are prescribed, and where a_{12} is a column vector with the same number of rows as A_{11}, a_{32} is also a column vector with the same number of rows as A_{33}, a_{21}^T is a row vector with the same number of columns as A_{11} and a_{23}^T is also a row vector with the same number of columns as A_{33}. Consider the completion of A

$$A_c = \begin{bmatrix} A_{11} & a_{12} & -a_{12}a_{23}^T \\ a_{21}^T & -1 & a_{23}^T \\ -a_{32}a_{21}^T & a_{32} & A_{33} \end{bmatrix}.$$

We are going to see that A_c is an N_0-matrix. Let α and β be the subsets of $N = \{1, 2, \ldots, n\}$ such that

$$A_c[\alpha] = \begin{bmatrix} A_{11} & a_{12} \\ a_{21}^T & -1 \end{bmatrix}, \quad \text{and} \quad A_c[\beta] = \begin{bmatrix} -1 & a_{23}^T \\ a_{32} & A_{33} \end{bmatrix},$$

and assume $|\alpha| = k$ (thus k is the index of the overlapping entry). Let $\gamma \subseteq N$. Then, there are two cases to consider:

(a) $k \in \gamma$. In this case,

$$\det A_c[\gamma] = (-1) \det A_c[\gamma \cap \alpha] \cdot \det A_c[\gamma \cap \beta] \leq 0.$$

(b) $k \notin \gamma$. We consider $\gamma = N \setminus \{k\}$. For another γ we proceed in analogous way. We are going to distinguish two cases.

(b1) A_{11} is non-singular. From $\det A_c[\alpha] \leq 0$ we obtain $\lambda = -a_{21}^T A_{11}^{-1} a_{12} \geq 1$ and since $\det A_c[\beta] \leq 0$ we have $\det(A_{33} + a_{32}a_{23}^T) \geq 0$.

Now, by applying Gauss elimination method we obtain $\det A_c[\gamma] = \det A_{11} \det(A_{33} + \lambda a_{32}a_{23}^T)$. To prove that $\det A_c[\gamma] \leq 0$ we need to show that $\det(A_{33} + \lambda a_{32}a_{23}^T) \geq 0$. By simplicity we denote $A_{33} = (c_{ij})_{i,j=1}^p$ and $a_{32}a_{23}^T = (b_{ij})_{i,j=1}^p$, where $p = n - k$.

It is easy to prove that, given $\rho \in \Re$

$$\det(A_{33} + \rho a_{32}a_{23}^T) = \det A_{33} + \rho M,$$

with

$$M = \det M_1 + \cdots \det M_p,$$

and where

$$M_1 = \begin{bmatrix} b_{11} & c_{12} & \cdots & c_{1p} \\ b_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ b_{p1} & c_{p2} & \cdots & c_{pp} \end{bmatrix}, \; \cdots, \; M_p = \begin{bmatrix} c_{11} & \cdots & c_{1p-1} & b_{1p} \\ c_{21} & \cdots & c_{2p-1} & b_{2p} \\ \vdots & & \vdots & \vdots \\ c_{p1} & \cdots & c_{pp-1} & b_{pp} \end{bmatrix}.$$

For $\rho = 1$, we have $\det(A_{33} + \rho a_{32}a_{23}^T) = \det A_{33} + \rho M = \det(A_{33} + a_{32}a_{23}^T) \geq 0$, then $M \geq -\det A_{33} \geq 0$. For $\rho = \lambda$,

$$\det(A_{33} + \lambda a_{32}a_{23}^T) \geq 0 \Longleftrightarrow \lambda M \geq -\det A_{33}.$$

Since $\lambda \geq 1$ and $M \geq 0$, we have $\lambda M \geq M \geq -\det A_{33}$. Therefore, $\det(A_{33} + \lambda a_{32}a_{23}^T) \geq 0$ and then $\det A_c[\gamma] \leq 0$.

(b2) A_{11} is singular. In this case, it is easy to see that

$$rank \begin{bmatrix} A_{11} \\ -a_{32}a_{21}^T \end{bmatrix} < k - 1,$$

and, then

$$rank \begin{bmatrix} A_{11} & -a_{12}a_{23}^T \\ -a_{32}a_{21}^T & A_{33} \end{bmatrix} < k - 1 + n - k = n - 1.$$

Therefore, $\det A_c[\gamma] = 0$. □

We can extend this result in the following way:

Theorem 1 *Let G be an undirected connected 1-chordal graph. Then any partial N_0-matrix, with no null main diagonal entries, the graph of whose specified entries is G, has an N_0-matrix completion.*

Proof: Let A be a partial N_0-matrix, the graph of whose specified entries is G. The proof is by induction on the number p, of maximal cliques in G. For $p = 2$ we obtain the desired completion by applying Proposition 5. Suppose that the result is true for a 1-chordal graph with $p - 1$ maximal cliques and we are going to prove it for p maximal cliques.

Let G_1 be the subgraph induced by two maximal cliques with a common vertex. By applying Proposition 5 to the submatrix A_1 of A, the graph of whose specified entries is G_1, and by replacing the obtained completion A_{1_c} in A, we obtain a partial N_0-matrix such that whose associated graph is 1-chordal with $p - 1$ maximal cliques. The induction hypothesis allows us to obtain the result. □

A partial matrix A is said to be *block diagonal* if A can be partitioned as

$$A = \begin{bmatrix} A_1 & X_{12} & \cdots & X_{1k} \\ X_{21} & A_2 & \cdots & X_{2k} \\ \vdots & \vdots & & \vdots \\ X_{k1} & X_{k2} & \cdots & A_k \end{bmatrix},$$

where X_{ij} are completely unspecified rectangular matrices and each A_i is a partial matrix, $i = 1, ..., k$.

Let A be a partial N_0-matrix whose associated graph G is non-connected. It is not difficult to prove the following theorem which establishes that if each submatrix associated with each connected component of G has an N_0-matrix completion, then so does the whole matrix A.

Theorem 2 *If a partial N_0-matrix A is permutation similar to a block diagonal partial matrix in which each diagonal block has an N_0-matrix completion, then A admits an N_0-matrix completion.*

From this result and taking into account that a partial matrix whose graph is non-connected is permutation similar to a block diagonal matrix, we can assume, without loss of generality, that the associated graph of a partial N_0-matrix is a connected graph.

The completion problem for partial N_0-matrices whose associated graph is p-chordal, $p > 1$, is still unresolved. We note here that any p-chordal graph, $p > 1$, contains, as an induced subgraph, a 2-chordal graph with four vertices. By left and right positive diagonal multiplication and by permutation similarity, we can assume, keeping in mind proposition 1, that a 4×4 partial N_0-matrix, the graph of whose prescribed entries is a 2-chordal graph, has the form

$$A = \begin{bmatrix} -1 & 1 & -a_{13} & x \\ a_{21} & -1 & 1 & -a_{24} \\ -a_{31} & a_{32} & -1 & 1 \\ y & -a_{42} & a_{43} & -1 \end{bmatrix},$$

with $a_{21}, a_{32}, a_{43} \geq 1$ and $a_{13}a_{31}, a_{24}a_{42} \geq 1$. It is easy to prove that

$$\det A = (a_{32}-1)xy - x\det A_0[\{2,3,4\}|\{1,2,3\}] - y\det A_0[\{1,2,3\}|\{2,3,4\}] + \det A_0 \quad (1)$$

From (1) we obtain sufficient conditions for the existence of the desired completion. Specifically, if $\det A_0[\{1,2,3\}|\{2,3,4\}] > 0$, we are going to see that A admits an N_0-matrix completion. We consider the following cases:

(i) Submatrix $A[\{2,3\}]$ is singular. Then, A has the form

$$A = \begin{bmatrix} -1 & 1 & -a_{13} & x \\ a_{21} & -1 & 1 & -a_{24} \\ -a_{31} & 1 & -1 & 1 \\ y & -a_{42} & a_{43} & -1 \end{bmatrix}.$$

Let $\bar{A}$ the partial N_0-matrix obtained by replacing entry x by c such that $0 < c < \min\{a_{13}, a_{24}\}$. Now the determinant of any principal submatrix containing position $(4,1)$ is a polynomial in y with negative leading coefficient. Therefore, there exists $M \in \Re$, $M > 0$, such that by completing position $(4,1)$ by every $d > M$ we obtain a N_0-matrix completion of A.

(ii) Submatrix $A[\{2,3\}]$ is nonsingular. Now consider the completion

$$A_c = \begin{bmatrix} -1 & 1 & -a_{13} & c \\ a_{21} & -1 & 1 & -a_{24} \\ -a_{31} & a_{32} & -1 & 1 \\ d & -a_{42} & a_{43} & -1 \end{bmatrix},$$

where $c, d \in \Re$ such that $d > 0$ and $0 < c < \min\{a_{13}, a_{24}, \dfrac{\det A_0[\{1,2,3\}|\{2,3,4\}]}{a_{32}-1}\}$. The determinant of any principal submatrix containing position $(4,1)$ is a polinomial in d with negative leading coefficient. Therefore, as in the previous case, there exists $M \in \Re$, $M > 0$ such that A_c is an N_0-matrix for $d > M$.

4 Cycles

In this section we are going to prove the existence of an N_0-completion for a partial N_0-matrix, with no null main diagonal entries, whose associated graph is a cycle.

Lemma 1 *Let A be an 4×4 partial N_0-matrix, with no null main diagonal entries, whose associated graph is a cycle. Then, there exists an N_0-matrix completion.*

Proof: Since the class of N_0-matrices is invariant under left and right positive diagonal multiplication, we may take all diagonal entries of A to be -1. We also know that A is permutationally similar to a partial N_0-matrix in PS_n. Hence we may assume that A has the following form:

$$A = \begin{bmatrix} -1 & 1 & x_{13} & a_{14} \\ a_{21} & -1 & 1 & x_{24} \\ x_{31} & a_{32} & -1 & 1 \\ a_{41} & x_{42} & a_{43} & -1 \end{bmatrix},$$

where $a_{21}, a_{32}, a_{43} \geq 1$ and $a_{14}a_{41} \geq 1$.

Let $\bar{A}$ the partial N_0-matrix obtained from A by replacing $x_{31} = -a_{32}$ and $x_{42} = -a_{41}$. It is easy to prove that there exists $x_{13} \geq 1/a_{32}$ such that $\det \bar{A}[\{1,3,4\}] \leq 0$ and there exists $x_{24} \geq 1/a_{41}$ such that $\det \bar{A}[\{2,3,4\}] \leq 0$. Since $\det \bar{A} = (a_{21} - 1) \det \bar{A}[\{1,3,4\}]$, we can conclude that there exists an N_0-matrix completion of A. $\square$

We extend this result in the following way.

Theorem 3 *Let A be an $n \times n$, $n \geq 4$, partial N_0-matrix, with no null main diagonal entries, whose associated graph is a cycle. Then, there exists an N_0-matrix completion.*

Proof: By left and right positive diagonal multiplication and by permutation similarity, we may assume that A has the following form:

$$A = \begin{bmatrix} -1 & 1 & x_{13} & \cdots & x_{1n-1} & (-1)^n a_{1n} \\ a_{21} & -1 & 1 & \cdots & x_{2n-1} & x_{2n} \\ x_{31} & a_{32} & -1 & \cdots & x_{3n-1} & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n-11} & x_{n-12} & x_{n-13} & \cdots & -1 & 1 \\ (-1)^n a_{n1} & x_{n2} & x_{n3} & \cdots & a_{nn-1} & -1 \end{bmatrix},$$

where $a_{1n}, a_{n1} > 0$ and $a_{ii-1} \geq 1, \ i = 2, 3, \ldots, n$.

The proof is by induction on n. For $n = 4$ see Lemma 1. Now, let A be an $n \times n$ matrix, $n > 4$. Consider the following partial N_0-matrix:

$$\bar{A} = \left[\begin{array}{ccccc|c} -1 & 1 & x_{13} & \cdots & (-1)^{n-1}a_{1n} & x_{1n} \\ a_{21} & -1 & 1 & \cdots & x_{2n-1} & x_{2n} \\ x_{31} & a_{32} & -1 & \cdots & x_{3n-1} & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ (-1)^{n-1}a_{n1} & x_{n-12} & x_{n-13} & \cdots & -1 & 1 \\ \hline x_{n1} & x_{n2} & x_{n3} & \cdots & a_{nn-1} & -1 \end{array}\right].$$

$\bar{A}[\{1,2,\ldots,n-1\}]$ is a partial N_0-matrix such that its associated graph is an $(n-1)$-cycle. By induction hypothesis there exists an N_0-matrix completion $\bar{A}[\{1,2,\ldots,n-1\}]_c$. Let $\hat{A}$ be the partial N_0-matrix obtained by replacing in $\bar{A}$ the completion $\bar{A}[\{1,2,\ldots,n-1\}]_c$.

Now, $\hat{A}$ is a partial N_0-matrix whose associated graph is 1-chordal with two maximal cliques, one of them with two vertices. By applying Proposition 4 to matrix $\hat{A}$ we obtain an N_0-matrix completion A_c of A. $\square$

In the last section we are going to consider the possibility that some main diagonal entries are zero.

5 Zero entries in the main diagonal

The results of Sections 3 and 4 do not hold if some main diagonal entries are zero, as we can see in the following example.

Example 3 (a) Consider the partial N_0-matrix

$$A = \begin{bmatrix} -1 & a_{12} & x_{13} & x_{14} \\ a_{21} & 0 & 0 & 0 \\ x_{31} & a_{32} & -1 & a_{34} \\ x_{41} & 0 & a_{43} & -1 \end{bmatrix}$$

with $a_{12}, a_{21} > 0$, whose associated graph is 1-chordal with two maximal cliques. Matrix A has no N_0-matrix completion since $\det A[\{1,2,4\}] = a_{12}a_{21} > 0$, for all x_{14} and x_{41}.

(b) Now, consider the partial N_0-matrix

$$B = \begin{bmatrix} 0 & 1 & x_{13} & 0 \\ 1 & -1 & 1 & x_{24} \\ x_{31} & 1 & 0 & 1 \\ 0 & x_{42} & 1 & -1 \end{bmatrix},$$

whose associated graph is a cycle. Matrix B has no N_0-matrix completion since $\det B[\{1,2,4\}] = 1 > 0$, for all x_{24} and x_{42}.

On the other hand, the completion problem for partial N_0-matrices, whose associated graph is p-chordal, $p > 1$, with some null main diagonal entries, has also a negative answer.

Example 4 The partial N_0-matrix

$$A = \begin{bmatrix} -1 & 1 & -1 & x_{14} \\ 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 1 \\ x_{41} & 0 & 2 & -1 \end{bmatrix},$$

the graph of whose specified entries is 2-chordal, has no N_0-matrix completion, since $\det A[\{1,2,4\}] = 1$, for all x_{14} and x_{41}.

Proposition 2 itself does not hold when some main diagonal entries are zero.

Example 5 The partial N_0-matrix

$$A = \begin{bmatrix} -1 & 0 & x_{13} \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

has no N_0-matrix completion, since $\det A = 1$, for all x_{13}.

In this context, the role of set S_n is taken by

$$wS_n = \{A = (a_{ij}) : \; a_{ij} = 0 \text{ or } sign(a_{ij}) = (-1)^{i+j+1}, \; \forall i, j\}.$$

Unfortunately an $n \times n$ N_0-matrix with some null diagonal entries is not, in general, diagonally similar to an N_0-matrix that belongs to wS_n, as the following example shows.

Example 6 Consider the N_0-matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}.$$

It is easy to observe that there is not a diagonal matrix D such that DAD^{-1} is an element of wS_n.

In order to obtain an N_0-matrix completion, when there are null elements in the main diagonal of the partial N_0-matrix, we introduce the following condition.

Definition 2 *Let A be a partial N_0-matrix. We say that A satisfies condition (ND) if*

$$a_{ii}a_{jj} = 0 \implies a_{ij}a_{ji} = 0,$$

when a_{ij} and a_{ji} are specified.

Proposition 6 *Let A be a combinatorially symmetric partial N_0-matrix, of size 3×3, with some null main diagonal entries. If A satisfies condition (ND), then there exists an N_0-matrix completion A_c of A.*

Proof: Since the class of N_0-matrices is invariant under left and right positive diagonal multiplication and under permutation similarity, we may assume, without loss of generality, that A has the form

$$A = \begin{bmatrix} -a_{11} & a_{12} & x \\ a_{21} & -a_{22} & a_{23} \\ y & a_{32} & -a_{33} \end{bmatrix},$$

where $a_{ii} \geq 0$, $i = 1, 2, 3$.

Consider the following cases:

(a) $a_{22} = 0$.
From condition (ND) we have $a_{12}a_{21} = 0$ and $a_{23}a_{32} = 0$. Therefore, $a_{12} = 0$ or $a_{21} = 0$ and $a_{23} = 0$ or $a_{32} = 0$. We can obtain in all cases values for x and y such that $xy \geq a_{11}a_{33}$ and $\det A = a_{32}a_{21}x + a_{12}a_{23}y \leq 0$.

(b) $a_{22} \neq 0$.
Then, $a_{11} = 0$ or $a_{33} = 0$. In both cases, the completion A_0 is an N_0-matrix.

The formulation of the problem in case of matrix A has more than two unspecified entries is reduced to this one. $\square$

However, condition (ND) is not a necessary condition, since matrix

$$A = \begin{bmatrix} -1 & 0 & x \\ 1 & 0 & 1 \\ y & 1 & -1 \end{bmatrix}$$

is a partial N_0-matrix, which does not satisfy condition (ND), but by taking $x = y = -1$ we obtain an N_0-matrix completion.

On the other hand, condition (ND) is not a sufficient condition for non-combinatorially symmetric partial N_0-matrix, as we see in the following example.

Example 7 Consider the partial N_0-matrix

$$A = \begin{bmatrix} -1 & 1 & x \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix},$$

which satisfies condition (ND) and, since $\det A = 1$, $\forall x$, it has no an N_0-matrix completion.

Motivated by the results obtained in sections $2-4$, we add a new restriction to our study: we will consider partial N_0-matrices belonging to the set PwS_n of $n \times n$ partial matrices (a_{ij}) such that, for all specified entry (i, j), $a_{ij} = 0$ or $sign(a_{ij}) = (-1)^{i+j+1}$. Note that we will implicitly analyze the completion problem for partial N_0-matrices that are permutation or diagonally similar to a partial N_0-matrix that belongs to PwS_n.

It is not difficult to prove the following result.

Proposition 7 *Let A be an 3×3 non-combinatorially symmetric partial N_0-matrix, with some null main diagonal entries, such that $A \in PwS_3$. If A satisfies condition (ND), then there exists an N_0-matrix completion of A.*

The following example shows that there exists a partial N_0-matrix that belong to PwS_4, which satisfies condition (ND), but has no an N_0-matrix completion. Therefore, we can not extend Proposition 6 and Proposition 7 for general n.

Example 8 Let A be the partial matrix

$$A = \begin{bmatrix} -1 & 0 & -1 & x \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ y & -2 & 0 & -1 \end{bmatrix}.$$

(We can take $y = 1$ for an example of a non-combinatorially symmetric partial matrix)

It is not difficult to verify that A is a partial N_0-matrix, $A \in PwS_4$ and it satisfies condition (ND). But A has no an N_0-matrix completion since $\det A = 2$, for all x and y.

Taking into account this example, a natural question arises: given an $n \times n$ partial N_0-matrix A, whose associated graph is an 1-chordal graph or a cycle, are conditions (ND) and $A \in PwS_n$ sufficient conditions in order to obtain the desired completion?

Proposition 8 *Let A be a 4×4 partial N_0-matrix belonging to PwS_4, with some null diagonal entries, the graph of whose specified entries is a cycle. If A satisfies condition (ND), then there exists an N_0-matrix completion of A.*

Proof: Let N_d be the number of null main diagonal entries. Consider the following cases:

(a) $N_d = 1$.

(a1) $a_{11} = 0$. Since the class of N_0-matrices is invariant under left and right positive diagonal multiplication and under permutation similarity, we may assume, without loss of generality, that matrix A has the form:

$$A = \begin{bmatrix} 0 & a_{12} & x_{13} & a_{14} \\ a_{21} & -1 & a_{23} & x_{24} \\ x_{31} & a_{32} & -1 & a_{34} \\ a_{41} & x_{42} & a_{43} & -1 \end{bmatrix},$$

with $a_{12}, a_{21}, a_{14}, a_{41} \geq 0$ and $a_{23}, a_{32}, a_{34}, a_{43} > 0$. We can prove that matrix

$$A_c = \begin{bmatrix} 0 & a_{12} & 0 & a_{14} \\ a_{21} & -1 & a_{23} & -a_{23}a_{34} \\ 0 & a_{32} & -1 & a_{34} \\ a_{41} & -a_{43}a_{32} & a_{43} & -1 \end{bmatrix}$$

is an N_0-matrix completion of A.

(a2) $a_{22} = 0$. In this case, matrix A has the form

$$A = \begin{bmatrix} -1 & a_{12} & x_{13} & a_{14} \\ a_{21} & 0 & a_{23} & x_{24} \\ x_{31} & a_{32} & -1 & a_{34} \\ a_{41} & x_{42} & a_{43} & -1 \end{bmatrix},$$

with $a_{12}, a_{21}, a_{23}, a_{32} \geq 0$ and $a_{14}, a_{41}, a_{34}, a_{43} > 0$. Matrix

$$A_c = \begin{bmatrix} -1 & a_{12} & -a_{14}a_{43} & a_{14} \\ a_{21} & 0 & a_{23} & 0 \\ -a_{34}a_{41} & a_{32} & -1 & a_{34} \\ a_{41} & 0 & a_{43} & -1 \end{bmatrix}$$

is an N_0-matrix completion of A.

(a3) $a_{33} = 0$. In this case matrix A is permutation similar to a matrix of type (a2).

(a4) $a_{44} = 0$. Analogously to case (a3), now matrix A is permutation similar to a matrix of type (a1).

(b) $N_d > 1$. Matrix A_0 is always an N_0-matrix completion of A. □

If, in the previous proposition, we leave out condition (ND) the result does not hold as we can see in Example 3, matrix B. In addition, if matrix A does not belong to PwS_4, in general, it has no an N_0-matrix completion. For example,

$$A = \begin{bmatrix} 0 & -1 & x_{13} & 0 \\ 0 & -1 & -1 & x_{24} \\ x_{31} & 0 & 0 & 1 \\ -2 & x_{24} & 0 & -1 \end{bmatrix}$$

is a partial N_0-matrix, that does not belong to PwS_4, satisfies condition (ND), and has no N_0-matrix completion since $\det A[\{1,2,4\}] \leq 0$ and $\det A[\{2,3,4\}] \leq 0$ if and only if $x_{24} < 0$ and $x_{24} > 0$, which is impossible.

Proposition 9 *Let A be a 4×4 partial N_0-matrix belonging to PwS_4, with some null main diagonal entries, the graph of whose specified entries is 1-chordal with two maximal cliques and null vertex separator. If A satisfies condition (ND), then there exists an N_0-matrix completion of A.*

Proof: Since the class of N_0-matrices is invariant under permutation similarity, we can suppose that the vertex separator is in position $(2,2)$. We consider the following cases:

(a) $a_{ii} \neq 0$, $i = 1,3,4$. We may assume, using right and left positive diagonal multiplication and permutation similarity, that matrix A has the form

$$A = \begin{bmatrix} -1 & a_{12} & x_{13} & x_{14} \\ a_{21} & 0 & a_{23} & -a_{24} \\ x_{31} & a_{32} & -1 & a_{34} \\ x_{41} & -a_{42} & a_{43} & -1 \end{bmatrix}.$$

By replacing $x_{13} = x_{31} = -1$, $x_{14} = a_{34}$ and $x_{41} = a_{43}$, we obtain an N_0-matrix completion of A.

(b) $a_{11} = 0$. Completion A_0 is an N_0-matrix.

(c) $a_{11} \neq 0$. Now we distinguish three possibilities: (c1) $a_{33} = a_{44} = 0$, (c2) $a_{33} = 0$ and $a_{44} \neq 0$, and (c3) $a_{33} \neq 0$ and $a_{44} = 0$. We obtain an N_0-matrix completion of A in each of them by analyzing a lot of cases, depending on each of the off-diagonal specified entries are or not zero. $\square$

Our study about the N_0-matrix completion problem, when the $n \times n$ partial N_0-matrix has some null diagonal entries, allows us to conjecture that, if we add condition (ND) as an hypothesis, the results showed in Sections 3 and 4 hold for partial N_0-matrices with some null diagonal entries. In addition, the mentioned condition can be left out in the case of 1-chordal graphs, when the vertex separator is non-zero.

References

[1] J.R.S. Blair, B. Peyton, An introduction to chordal graphs and clique trees, *The IMA volumes in Mathematics and its Applications*, Vol. 56, Springer, New

York, 1993, pp. 1-31.

[2] K. Inada, The production coefficient matrix and the Stolper-Samuelson condition, *Econometrica*, **39** (1971), 219-239.

[3] C. Mendes, Juan R. Torregrosa, Ana M. Urbano, N-matrix completion problem, *Linear Algebra and its Applications*, **372** (2003), 111-125.

[4] S.R. Mohan, R. Sridhar, On characterizing N-matrices using linear complementarity, *Linear Algebra and its Applications*, **160** (1992), 231-245.

[5] S.R. Mohan, Degeneracy in linear complementarity problems: a survey, *Annals of Operations Research*, **46-47** (1993), 179-194.

[6] S.R. Paranjape, Simple proofs for the infinite divisibility of multivariate gamma distributions, *Sankhya Ser. A*, **40** (1978), 393-398.