ON A CONSTRAINED REACTION-DIFFUSION SYSTEM RELATED TO MULTIPHASE PROBLEMS

José-Francisco Rodrigues

Universidade de Lisboa/CMAF
Av. Prof. Gama Pinto 2
1649–003 Lisboa, Portugal

Lisa Santos

Universidade do Minho/CMat
Campus de Gualtar
4710 - 057 Braga, Portugal

Abstract. We solve and characterize the Lagrange multipliers of a reaction-diffusion system in the Gibbs simplex of $\mathbb{R}^{N+1}$ by considering strong solutions of a system of parabolic variational inequalities in $\mathbb{R}^N$. Exploring properties of the two obstacles evolution problem, we obtain and approximate a N-system involving the characteristic functions of the saturated and/or degenerated phases in the nonlinear reaction terms. We also show continuous dependence results and we establish sufficient conditions of non-degeneracy for the stability of those phase subregions.

Dedicated to MASAYASU MIMURA on the occasion of its 65th birthday

1. Introduction

This paper is motivated by the vector-valued reaction-diffusion equation

$$\partial_t \boldsymbol{U} - \Delta \boldsymbol{U} = \boldsymbol{F}(x,t,\boldsymbol{U}), \qquad \text{in } Q, \tag{1}$$

for $\boldsymbol{U} = \boldsymbol{U}(x,t)$, defined from $Q = \Omega \times (0,T)$ into $\mathbb{R}^{N+1}$, with homogeneous Neumann condition on $\partial\Omega \times (0,T)$, where Ω is a bounded domain of $\mathbb{R}^n$ and $T > 0$ is arbitrary. We are interested in the case when every component $u_i = u_i(x,t)$ is nonnegative and the system is subject to the multiphase non-voids condition with $\boldsymbol{J} = (1,\dots,1) \in \mathbb{R}^{N+1}$:

$$\boldsymbol{U} \cdot \boldsymbol{J} = \sum_{j=1}^{N+1} u_j = 1 \qquad \text{in } Q. \tag{2}$$

From the equation (1) it is clear that the constraint (2) implies $\boldsymbol{F}(x,t,\boldsymbol{U}) \cdot \boldsymbol{J} = 0$ in Q and so the reaction vector $\boldsymbol{F}$ should satisfy the necessary and very restrictive

1991 *Mathematics Subject Classification.* Primary: 35K57, 35K85; Secondary: 35R35, 74N25, 91A99, 92D99.

Key words and phrases. Reaction-diffusion systems, multiphase problems, parabolic variational inequalities, evolutionary game dynamics.

Research partially supported by FCT Project POCI/MAT/57546/2004.

condition

$$(3)\quad F_{N+1}(x,t,V) = -\sum_{j=1}^{N} F_j(x,t,V) \quad \text{in } Q,$$

$$\forall\, V = (v_1,\dots,v_n, 1-\sum_{j=1}^{N} v_j),\ 0 \le v_i \le 1.$$

For instance, in replicator dynamics describing the evolution of certain frequencies in a population, one possible definition of the reaction term with this compatibility condition consists in choosing

$$(4)\quad F_i(x,t,V) = v_i[\phi_i(x,t,V) - \sum_{j=i}^{N+1} v_j\phi_j(x,t,V)] \qquad \text{in } Q, \qquad i=1,...,N+1,$$

where v_i represents the i-frequency of the population and ϕ_i the respective fitness (see, for instance, [12] and [13]), the constraint (2) is essential to describe mixed strategies in evolutionary game theory in spatially homogeneous population dynamics (see [20] and its references) or to model the non-voids condition in biological tissue growing [17, 16]. In phase fields models, the condition (2) arises naturally in simulation of multiphase flows [15], multiphase systems with diffuse phase boundaries, as in solidification of alloys or in grain boundary motion (see [11] or [3]) and in multicomponent mixtures [9] and [8].

Of course, in the case (3), in particular, if $\boldsymbol{F} = \boldsymbol{0}$, the problem becomes a simple one if the initial data $\boldsymbol{U}(0) = \boldsymbol{U}_0$ also satisfies the constraint (2). However the situation is entirely different in the general case of non trivial reactions, specially in multiphase problems where at least one phase "i" in a subregion of Q is absent (i.e. $u_i = 0$), or fulfils another subregion (when $u_i = 1$).

Instead of solving the system (1) in the Gibbs (N+1)-simplex

$$\Psi = \{(v_1,\dots,v_{N+1}) \in R^{N+1} : \sum_{j=1}^{N+1} v_j = 1 \text{ and } v_i \ge 0,\ i=1,\dots,N+1\},$$

we shall replace this problem by the study of a unilateral problem for the vector field of the first N components $\boldsymbol{u} = (u_1,\dots,u_N)$ of $\boldsymbol{U}$, with the $N+1$ convex constraints

$$(5)\qquad \sum_{i=j}^{N} u_j \le 1 \qquad \text{and} \qquad u_i \ge 0 \qquad \text{in } Q, \qquad i=1,\dots,N.$$

This corresponds to solve the system of parabolic variational inequalities, at each time $t \in (0,T)$,

$$(6)\quad \boldsymbol{u}(t) \in \mathbb{K}: \qquad \int_\Omega \partial_t \boldsymbol{u}(t)\cdot\big(\boldsymbol{v}-\boldsymbol{u}(t)\big) + \int_\Omega \nabla\boldsymbol{u}(t)\cdot\nabla(\boldsymbol{v}-\boldsymbol{u}(t))$$
$$\ge \int_\Omega \boldsymbol{f}(\boldsymbol{u}(t))\cdot\big(\boldsymbol{v}-\boldsymbol{u}(t)\big), \qquad \forall \boldsymbol{v}\in\mathbb{K},$$

under the initial condition

$$(7)\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0 = (u_{01},\dots,u_{0N}) \in \mathbb{K}.$$

Here $\mathbb{K}$ denotes the convex subset of the Sobolev space $H^1(\Omega)^N$ defined by

$$\mathbb{K} = \{\boldsymbol{v} \in H^1(\Omega)^N : \sum_{j=1}^{N} v_j \leq 1,\ v_i \geq 0,\ i = 1, \dots, N,\ \text{in } \Omega\}, \tag{8}$$

where $\boldsymbol{v} = (v_1, \dots, v_N)$.

The reaction term may have a general form $f_i(\boldsymbol{u}) = f_i(x, t, \boldsymbol{U}(x,t))$, $i = 1, \dots, N$, with $(x,t) \in Q$ and $\boldsymbol{U} = \big(u_1, \dots, u_N, 1 - \sum_{j=1}^{N} u_j\big)$. We denote $\partial_t = \dfrac{\partial}{\partial t}$ and $\nabla = \Big(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}\Big)$.

The main part of this work is the analysis of the new unilateral problem (6)-(7) under general assumptions on $\boldsymbol{f}$: only continuity on $\boldsymbol{u}$ and integrability in $(x,t) \in Q$. In particular, we prove that its solution $\boldsymbol{u} = \boldsymbol{u}(x,t)$ is such that each component u_i satisfies a double obstacle problem

$$0 \leq u_i \leq 1 - \sum_{j \neq i} u_j \qquad \text{in } Q, \qquad i = 1, \dots, N, \tag{9}$$

where $\sum_{j \neq i} u_j$ denotes the sum of all $N-1$ components but u_i. In fact, u_i is also the solution of a reaction-diffusion system in the form

$$\begin{aligned}\partial_t u_i - \Delta u_i = f_i(\boldsymbol{u}) &+ f_i^-(\boldsymbol{u})\chi_{\{u_i=0\}} \\ &- \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k}\big(f_{i_1}(\boldsymbol{u}) + \dots + f_{i_k}(\boldsymbol{u})\big)^+ \chi_{i_i \dots i_k}, \qquad \text{in } Q.\end{aligned} \tag{10}$$

Here $\sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}}$ denotes the summation over all the subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, N\}$ to which i belongs, in particular, k varies from 1 to N. We also denote $g^+ = g \vee 0$ and $g^- = -(g \wedge 0)$ the positive and negative parts of a scalar function $g = g^+ - g^-$, χ_A the characteristic function of the set A, (i.e., $\chi_A = 1$ in A and $\chi_A = 0$ in $Q \setminus A$) and $\chi_{i_1 \dots i_k}$ the characteristic function of the set

$$I_{i_1 \dots i_k} = \big\{(x,t) \in Q : \big(u_{i_1} + \dots + u_{i_k}\big)(x,t) = 1,\ u_{i_j}(x,t) > 0,\ j = 1, \dots, k\big\},$$

where $k \in \{1, \dots, N\}$.

In particular $\{u_i = 1\} = \bigcap_{j \neq i} \{u_j = 0\}$, i.e., one component is fully saturated if and only if the others are absent. Hence from (10) we see that, in general, the respective reaction terms are coupled not only through the semilinear term $\boldsymbol{f}(\boldsymbol{u})$ but also through the characteristic functions of the saturation sets $I_{i_1 \dots i_k}$.

In this way, by setting for $i = 1, \dots, N$,

$$F_i(\boldsymbol{U}) = f_i(\boldsymbol{u}) + f_i^-(\boldsymbol{u})\chi_{\{u_i=0\}} - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k}\big(f_{i_1}(\boldsymbol{u}) + \dots + f_{i_k}(\boldsymbol{u})\big)^+ \chi_{i_i \dots i_k},$$

with $\boldsymbol{U} = (\boldsymbol{u}, 1 - \sum_{j=1}^{N} u_j)$, we can solve the system (1) under the constraint (2) and identify the respective Lagrange multipliers $h_i \equiv F_i(\boldsymbol{U}) - f_i(\boldsymbol{U})$ in a precise form.

To illustrate the meaning of the system (10), that contains $2^N - 1 + N$ characteristic functions, in general, we may consider the cases $N = 1, 2$ or 3. Denoting, for simplicity, $f_i = f_i(\boldsymbol{u})$, $\chi_i = \chi_{\{u_i=1\}}$, we may write the Lagrange multipliers as

$$h_1 = f_1^- \chi_{\{u_1=0\}} - f_1^+ \chi_1 - \tfrac{1}{2}(f_1 + f_2)^+ \chi_{12} - \tfrac{1}{2}(f_1 + f_3)^+ \chi_{13} - \tfrac{1}{3}(f_1 + f_2 + f_3)^+ \chi_{123}$$

$$h_2 = f_2^- \chi_{\{u_2=0\}} - f_2^+ \chi_2 - \tfrac{1}{2}(f_1 + f_2)^+ \chi_{12} - \tfrac{1}{2}(f_2 + f_3)^+ \chi_{23} - \tfrac{1}{3}(f_1 + f_2 + f_3)^+ \chi_{123}$$

$$h_3 = f_3^- \chi_{\{u_3=0\}} - f_3^+ \chi_3 - \tfrac{1}{2}(f_1 + f_3)^+ \chi_{13} - \tfrac{1}{2}(f_2 + f_3)^+ \chi_{23} - \tfrac{1}{3}(f_1 + f_2 + f_3)^+ \chi_{123}$$

Ignoring the third equation and all the terms involving the third component, we may obtain the case $N = 2$. The first two terms of the right hand side of the first equation correspond, in the case $N = 1$, to the scalar two obstacles problem that has been proposed for phase separations in [4, 5].

The mathematical treatment of this unilateral system is done in the following three sections. In section 2, we consider the semilinear approximation of the unique solution of (6)-(7) in the case of the reaction $\boldsymbol{f}$ is in $L^2(Q)^N$ and independent of the solution. Although there exists a large literature on parabolic variational inequalities (see, for instance, [18], [6], [14], [7] or [10]), the direct approach of the bounded penalization used for the two obstacles problem in [23] (see also [21]), extended here for the system (10), allows the use of monotone methods. This yields a direct way of obtaining Lewy-Stamppachia inequalities (26), obtained first by [7] for parabolic problems, implying the $W^{2,1}_p$ and Hölder regularity for the solution to (6). Similar results for the N-membranes stationary problem have been obtained in [1, 2]. We note in our case the simplification due to homogeneous Neumann condition.

In section 3, we extend the existence result to general nonlinear reaction $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{u})$ taking values in $L^1(Q)^N$. Here we explore the fact that the convex set (8) lies in the unit disc and we extend the direct technique of [22]. We show also a continuous dependence result and, in the case of $\lambda I - \boldsymbol{f}$ being monotone non-decreasing, in particular if $\boldsymbol{f}$ is Lipschitz continuous in $\boldsymbol{u}$, also the uniqueness of solution and their strong approximation by the penalized solutions.

Finally, in the last section, we characterize the solution of the variational inequality (6) as solutions of the reaction-diffusion system (10), by extending some remarks of [24] to the two obstacles parabolic problem. We also show that

$$\{u_i = 0\} \subset \{f_i(\boldsymbol{u}) \leq 0\} \qquad \text{and} \qquad I_{i_1 \dots i_k} \subset \Big\{ \sum_{j=1}^{k} f_{i_j}(\boldsymbol{u}) \geq 0 \Big\}$$

a.e. in Q, for $1 \leq i \leq N$, $1 \leq i_1 < \cdots < i_k \leq N$, $\forall\, k = 1, \dots, N$ and we can modify the system (10) (see (77)) and show that the a.e. pointwise nondegeneracy assumptions

$$\sum_{j=1}^{k} f_{i_j}(\boldsymbol{u}) \neq 0, \qquad 1 \leq i_1 < \cdots < i_k \leq N, \quad k = 1, \dots, N,$$

are sufficient conditions for the local stability of the characteristic functions $\chi_{\{u_i=0\}}$ and $\chi_{i_1 \dots i_k}$ with respect to the perturbation of the nonlinear reaction terms $\boldsymbol{f}$.

2. Approximation of strong solutions by semilinear problems

In this section we consider the case where $\boldsymbol{f} = (f_1, \dots, f_N)$ depends only on (x,t) and is given in $L^2(Q)^N$.

To prove existence of solution of the variational inequality (6)-(7), we consider a family of approximating semilinear systems of equations. We define, for each $\varepsilon > 0$, $\theta_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\theta_\varepsilon(s) = \begin{cases} 0 & \text{if } s \geq 0 \\ s/\varepsilon & \text{if } -\varepsilon < s < 0 \\ -1 & \text{if } s \leq -\varepsilon, \end{cases} \tag{11}$$

and we denote

$$P\boldsymbol{u} = \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} = (Pu_1, \dots, Pu_N),$$

where $\partial_t \boldsymbol{u} = (\partial_t u_1, \dots, \partial_t u_N)$ and $\Delta \boldsymbol{u} = (\Delta u_1, \dots, \Delta u_N)$. We also denote $Pu_i = \partial_t u_i - \Delta u_i$, $i = 1, \dots, N$. The approximating problems are given by the following weakly coupled parabolic system with Neumann condition

$$Pu_i^\varepsilon + f_i^- \theta_\varepsilon(u_i^\varepsilon) - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k}\big(f_{i_1} + \dots + f_{i_k}\big)^+ \theta_\varepsilon(1 - u^\varepsilon_{i_1 \dots i_k}) = f_i \qquad \text{in } Q, \tag{12}$$

$$\frac{\partial u_i^\varepsilon}{\partial \boldsymbol{n}} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{13}$$

$$u_i^\varepsilon(0) = u_{0i} \qquad \text{in } \Omega, \qquad\qquad (i = 1, \dots, N) \tag{14}$$

where $\dfrac{\partial}{\partial \boldsymbol{n}}$ is the outward normal derivative on $\partial\Omega \times (0,T)$, the meaning of $\displaystyle\sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}}$ was explained in the introduction and

$$\forall\, \boldsymbol{v} = (v_1, \dots, v_N) \;\; \forall \{i_1, \dots, i_k\} \subseteq \{1, \dots, N\} \qquad v_{i_1 \dots i_k} = v_{i_1} + \dots + v_{i_k}. \tag{15}$$

Defining the penalization operator Θ_ε by

$$\Theta_\varepsilon \boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^N \Big[f_i^- \theta_\varepsilon(u_i) - \sum_{\substack{1 \leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k}\big(f_{i_1} + \dots + f_{i_k}\big)^+ \theta_\varepsilon(1 - u_{i_1 \dots i_k}) \Big] v_i \tag{16}$$

$$= \sum_{i=1}^N f_i^- \theta_\varepsilon(u_i) v_i - \sum_{1 \leq i_1 < \dots < i_k \leq N} \frac{1}{k}\big(f_{i_1} + \dots + f_{i_k}\big)^+ \theta_\varepsilon(1 - u_{i_1 \dots i_k}) v_{i_1 \dots i_k}, \tag{17}$$

we formulate (12)-(13) in variational form for a.e. $t \in (0,T)$,

$$\int_\Omega \partial_t \boldsymbol{u}^\varepsilon(t) \cdot \boldsymbol{v} + \int_\Omega \nabla \boldsymbol{u}^\varepsilon(t) \cdot \nabla \boldsymbol{v} + \int_\Omega \Theta_\varepsilon(\boldsymbol{u}^\varepsilon(t)) \cdot \boldsymbol{v} = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v}, \qquad \forall \boldsymbol{v} \in H^1(\Omega)^N, \tag{18}$$

associated with the initial condition (14).

Proposition 2.1. *Assuming that*

$$\boldsymbol{f}=(f_1,\dots,f_N)\in L^2(Q)^N \qquad \textit{and} \qquad \boldsymbol{u}_0\in\mathbb{K}, \tag{19}$$

the problem (18)-(14) *has a unique solution*

$$\boldsymbol{u}^\varepsilon\in H^1(0,T;L^2(\Omega)^N)\cap L^\infty(0,T;H^1(\Omega)^N).$$

Proof. We begin by proving the monotonicity of the penalization operator Θ_ε.

In fact, recalling that θ_ε is monotone nondecreasing and the definition (15) we have

$$\begin{aligned}
&\big(\Theta_\varepsilon\boldsymbol{u}-\Theta_\varepsilon\boldsymbol{v}\big)\cdot(\boldsymbol{u}-\boldsymbol{v})\\
&\quad=\sum_{i=1}^N f_i^-\big(\theta_\varepsilon(u_i)-\theta_\varepsilon(v_i)\big)(u_i-v_i)\\
&\quad-\sum_{1\le i_1<\dots<i_k\le N}\frac1k\big(f_{i_1}+\cdots f_{i_k}\big)^+\big(\theta_\varepsilon(1-u_{i_1\dots i_k})-\theta_\varepsilon(1-v_{i_1\dots i_k})\big)(u_{i_1\dots i_k}-v_{i_1\dots i_k}),\\
&\quad\ge 0,
\end{aligned}$$

since f_j^- and $\big(f_{i_1}+\cdots f_{i_k}\big)^+$ are nonnegative functions.

The existence and uniqueness of solution $\boldsymbol{u}^\varepsilon\in L^2(0,T;H^1(\Omega)^N)$ is immediate by applying the theory of monotone operators ([18], [26])).

Setting $\boldsymbol{v}=(u_1^\varepsilon,\dots,u_N^\varepsilon)$ in the approximating problem (18) and integrating in time, letting

$$g_i^\varepsilon=Pu_i^\varepsilon=f_i-f_i^-\theta_\varepsilon(u_i^\varepsilon)+\sum_{\substack{1\le i_1<\dots<i_k\le N\\ i\in\{i_1,\dots,i_k\}}}\frac1k\big(f_{i_1}+\cdots+f_{i_k}\big)^+\theta_\varepsilon(1-u^\varepsilon_{i_1\dots i_k}),$$

which is bounded in $L^2(Q)$ independently of ε, we obtain that, for every $0<t<T$, with $Q_t=\Omega\times(0,t)$,

$$\frac12\int_\Omega|\boldsymbol{u}^\varepsilon(t)|^2+\int_{Q_t}|\nabla\boldsymbol{u}^\varepsilon|^2\le\frac12\int_\Omega|\boldsymbol{u}_0|^2+\frac12\int_{Q_t}|\boldsymbol{g}^\varepsilon|^2+\frac12\int_{Q_t}|\boldsymbol{u}^\varepsilon|^2.$$

The Grownwall inequality yields the uniform boundedeness (in ε) of $\boldsymbol{u}^\varepsilon$, first in $L^\infty(0,T;L^2(\Omega)^N))$ and afterwards also in $L^2(0,T;H^1(\Omega)^N)$.

Letting, formally, $\boldsymbol{v}=\partial_t\boldsymbol{u}^\varepsilon$ in (18) (in fact in the respective Faedo-Galerkin approximation) and integrating in time, we get

$$\int_{Q_t}\big|\partial_t\boldsymbol{u}^\varepsilon\big|^2+\int_\Omega|\nabla\boldsymbol{u}^\varepsilon(t)|^2\le\int_{Q_t}|\boldsymbol{g}^\varepsilon|^2+\int_\Omega|\nabla\boldsymbol{u}_0|^2$$

and so $\partial_t\boldsymbol{u}_\varepsilon$ is also bounded in $L^2(Q)^N$ and $\nabla\boldsymbol{u}^\varepsilon$ in $L^\infty(0,T;L^2(\Omega)^N)$. Therefore

$$\{\boldsymbol{u}^\varepsilon\}_{\varepsilon>0}\ \text{ is bounded in }\ H^1(0,T;L^2(\Omega)^N))\cap L^\infty(0,T;H^1(\Omega)^N). \tag{20}$$

□

Proposition 2.2. *Assuming* (19), *the solution* $\boldsymbol{u}^\varepsilon$ *of the problem* (18)-(14) *satisfies*

$$u_i^\varepsilon\ge-\varepsilon,\qquad i=1,\dots,N,\qquad\qquad \sum_{i=1}^N u_i^\varepsilon\le1+\varepsilon. \tag{21}$$

Proof. In fact, we are going to prove the following more general set of inequalities

$$u_i^\varepsilon \geq -\varepsilon, \quad i = 1, \ldots, N, \qquad \text{and} \qquad u_{i_1 \ldots i_r}^\varepsilon \leq 1 + \varepsilon, \quad \forall\, 1 \leq i_1 < \ldots < i_r \leq N$$

and the proof of the right hand side inequalities will be done by induction on r.

Let us prove the case $r = 1$, i.e., $u_i^\varepsilon \leq 1 + \varepsilon$, for all $i \in \{1, \ldots, N\}$. Multiplying the i-th equation of the approximating system (12) by $(u_i^\varepsilon - (1+\varepsilon))^+$ and integrating over $Q_t = \Omega \times (0, t)$, we have

$$\int_{Q_t} \partial_t u_i^\varepsilon (u_i^\varepsilon - (1+\varepsilon))^+ + \int_{Q_t} \nabla u_i^\varepsilon \cdot \nabla (u_i^\varepsilon - (1+\varepsilon))^+ = \int_{Q_t} \Big[f_i - f_i^- \theta_\varepsilon(u_i^\varepsilon) \\ + \sum_{\substack{1 \leq\, i_1 < \ldots <\, i_k \leq\, N \\ i\, \in\, \{i_1, \ldots, i_k\}}} \frac{1}{k} \big(f_{i_1} + \cdots + f_{i_k}\big)^+ \theta_\varepsilon(1 - u_{i_1 \ldots i_k}^\varepsilon) \Big] (u_i^\varepsilon - (1+\varepsilon))^+$$

Recalling that $-1 \leq \theta_\varepsilon \leq 0$ and that, in the set $\{u_i^\varepsilon > 1 + \varepsilon\}$, we have $\theta_\varepsilon(u_i^\varepsilon) = 0$ and $\theta_\varepsilon(1 - u_i^\varepsilon) = -1$ we get

(22)
$$\frac{1}{2} \int_\Omega |(u_i^\varepsilon - (1+\varepsilon))^+(t)|^2 + \int_Q |\nabla (u_i^\varepsilon - (1+\varepsilon))^+|^2 \leq \int_Q (f_i - f_i^+)(u_i^\varepsilon - (1+\varepsilon))^+ \leq 0,$$

so $(u_i^\varepsilon - (1+\varepsilon))^+ \equiv 0$, i.e. $u_i^\varepsilon \leq 1 + \varepsilon$.

Assuming we have proved that $u_{i_1 \ldots i_r}^\varepsilon \leq 1 + \varepsilon$, we are going to show that $u_{i_1 \ldots i_r i_{r+1}}^\varepsilon \leq 1 + \varepsilon$.

We multiply the equations i_j, $j = 1, \ldots, r+1$, by $(u_{i_1 \ldots i_r i_{r+1}}^\varepsilon - (1+\varepsilon))^+$, sum from 1 to $r+1$ and integrate over Q_t. We obtain

$$\int_{Q_t} P u_{i_1 \ldots i_r i_{r+1}}^\varepsilon (u_{i_1 \ldots i_r i_{r+1}}^\varepsilon - (1+\varepsilon))^+ = \int_{Q_t} \Big[\sum_{j=1}^{r+1} f_{i_j} - \sum_{j=1}^{r+1} f_{i_j}^- \theta_\varepsilon(u_{i_j}^\varepsilon) \\ + \sum_{j=1}^{r+1} \sum_{\substack{1 \leq\, i_1 < \ldots <\, i_k \leq\, N \\ i_j\, \in\, \{i_1, \ldots, i_k\}}} \frac{1}{k} \big(f_{i_1} + \cdots + f_{i_k}\big)^+ \theta_\varepsilon(1 - u_{i_1 \ldots i_k}^\varepsilon) \Big] (u_{i_1 \ldots i_r i_{r+1}}^\varepsilon - (1+\varepsilon))^+.$$

Observe that, in the set $\{u_{i_1 \ldots i_r i_{r+1}}^\varepsilon > 1 + \varepsilon\}$ we have $u_{i_j}^\varepsilon \geq 0$, for $j = 1, \ldots, r+1$, since, by induction, $u_{l_1 \ldots l_r}^\varepsilon = u_{i_1}^\varepsilon + \cdots + u_{i_{r+1}}^\varepsilon - u_{i_j}^\varepsilon \leq 1 + \varepsilon$. So, in that set, $\theta_\varepsilon(u_{i_j}^\varepsilon) = 0$ and, on the other hand, $\theta_\varepsilon(1 - u_{i_1 \ldots i_r i_{r+1}}^\varepsilon) = -1$. The induction conclusion follows from

$$\int_\Omega |(u_{i_1 \ldots i_r i_{r+1}}^\varepsilon - (1+\varepsilon))^+(t)|^2 + \int_{Q_t} |\nabla (u_{i_1 \ldots i_r i_{r+1}}^\varepsilon - (1+\varepsilon))^+|^2 \\ \leq \int_{Q_t} \Big[\sum_{j=1}^{r+1} f_{i_j} - (r+1) \frac{1}{r+1} \big(f_{i_1} + \cdots + f_{i_{r+1}}\big)^+ \Big] (u_{i_1 \ldots i_r i_{r+1}}^\varepsilon - (1+\varepsilon))^+ \leq 0.$$

To prove that $u_i^\varepsilon \geq -\varepsilon$, we multiply the i-th equation of (12) by $(-u_i^\varepsilon - \varepsilon)^+$, obtaining

$$\frac{1}{2}\int_\Omega |(-u_i^\varepsilon - \varepsilon)^+(t)|^2 + \int_Q |\nabla(-u_i^\varepsilon - \varepsilon)^+|^2 = \int_Q \Big[- f_i + f_i^- \theta_\varepsilon(u_i^\varepsilon)$$
$$- \sum_{\substack{1\leq i_1 < \ldots < i_k \leq N \\ i \in \{i_1,\ldots,i_k\}}} \frac{1}{k}\big(f_{i_1} + \cdots + f_{i_k}\big)^+ \theta_\varepsilon(1 - u_{i_1\ldots i_k}^\varepsilon)\Big](-u_i^\varepsilon - \varepsilon)^+.$$

Let $J_{k,i} = \{i_1,\ldots,i_k\}\setminus\{i\}$ and denote the elements of $J_{k,i}$ by $j_1,\ldots,j_{k-1}$. Since, in the set $\{(-u_i^\varepsilon - \varepsilon)^+ > 0\} = \{u_i^\varepsilon < -\varepsilon\}$, we have $1 - u_{i_1\ldots i_k}^\varepsilon = 1 - u_{j_1\ldots j_{k-1}}^\varepsilon - u_i^\varepsilon > 0$ (recall that $u_{j_1\ldots j_{k-1}}^\varepsilon \leq 1+\varepsilon$), so

$$\frac{1}{2}\int_\Omega |(-u_i^\varepsilon - \varepsilon)^+(t)|^2 + \int_Q |\nabla(-u_i^\varepsilon - \varepsilon)^+|^2$$
$$\leq \int_Q \Big[- f_i - f_i^- \Big](-u_i^\varepsilon - \varepsilon)^+ = \int_Q -f_i^+(-u_i^\varepsilon - \varepsilon)^+ \leq 0,$$

that implies $(-u_i^\varepsilon - \varepsilon)^+ = 0$, or $u_i^\varepsilon \geq -\varepsilon$. □

Theorem 2.3. *Assuming* (19)*, the variational inequality* (6)-(7) *has a unique solution* $\boldsymbol{u}$ *such that*

$$\boldsymbol{u} \in H^1(0,T;L^2(\Omega)^N) \cap L^\infty(0,T;H^1(\Omega)^N) \tag{23}$$

and

$$P\boldsymbol{u} \in L^2(Q)^N. \tag{24}$$

Proof. Let $\boldsymbol{u}^\varepsilon$ be the solution of the problem (18). Using the uniform estimates (in ε) obtained in (20), by compactness, we know there exists $\boldsymbol{u}$ such that

$$\boldsymbol{u}^\varepsilon \xrightarrow[\varepsilon]{} \boldsymbol{u} \qquad \text{in } L^2(Q)^N \text{ strong},$$

$$\boldsymbol{u}^\varepsilon \xrightharpoonup[\varepsilon]{} \boldsymbol{u} \qquad \text{in } L^\infty(0,T;H^1(\Omega)^N) \text{ weak-}*,$$

$$\partial_t\boldsymbol{u}^\varepsilon \xrightharpoonup[\varepsilon]{} \partial_t\boldsymbol{u} \qquad \text{and} \qquad P\boldsymbol{u}^\varepsilon \rightharpoonup P\boldsymbol{u} \qquad \text{in } L^2(Q)^N \text{ weak}.$$

We have $\boldsymbol{u}(t) \in \mathbb{K}$, for a.e. $t \in [0,T]$, because $\boldsymbol{u}^\varepsilon$ satisfies the inequalities (21).

Given $\boldsymbol{v} \in L^2(0,T;\mathbb{K})$, set $\boldsymbol{v}(t) - \boldsymbol{u}^\varepsilon(t)$ in (18) and integrate in time. Then

$$\int_Q \partial_t\boldsymbol{u}^\varepsilon \cdot (\boldsymbol{v} - \boldsymbol{u}^\varepsilon) + \int_Q \nabla\boldsymbol{u}^\varepsilon \cdot \nabla(\boldsymbol{v} - \boldsymbol{u}^\varepsilon) \geq \int_Q \boldsymbol{f}^\varepsilon \cdot (\boldsymbol{v} - \boldsymbol{u}^\varepsilon),$$

since $\displaystyle\int_Q \big(\Theta_\varepsilon(\boldsymbol{u}^\varepsilon) - \Theta_\varepsilon(\boldsymbol{v})\big)\cdot(\boldsymbol{v} - \boldsymbol{u}^\varepsilon) \leq 0$ and $\Theta_\varepsilon(\boldsymbol{v}(t)) = 0$ if $\boldsymbol{v}(t) \in \mathbb{K}$. Passing to the limit when $\varepsilon \to 0$ and noting that

$$\liminf_{\varepsilon\to 0} \int_Q \big(\partial_t\boldsymbol{u}^\varepsilon \cdot \boldsymbol{u}^\varepsilon + \nabla\boldsymbol{u}^\varepsilon \cdot \nabla\boldsymbol{u}^\varepsilon\big) \geq \int_Q \big(\partial_t\boldsymbol{u} \cdot \boldsymbol{u} + \nabla\boldsymbol{u} \cdot \nabla\boldsymbol{u}\big),$$

we find that $\boldsymbol{u}$ satisfies (7) and

$$\int_Q \partial_t\boldsymbol{u} \cdot (\boldsymbol{v} - \boldsymbol{u}) + \int_Q \nabla\boldsymbol{u} \cdot \nabla(\boldsymbol{v} - \boldsymbol{u}) \geq \int_Q \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}), \qquad \forall \boldsymbol{v} \in L^2(0,T;\mathbb{K}), \tag{25}$$

which is easily seen to be equivalent to (6). The uniqueness is immediate. □

We remark that no regularity of the boundary $\partial\Omega$ has been required in (18) and, in fact, the Neumann boundary condition (13) is only formal. In the proof of Theorem 2.3 we have used the compactness of the sequence $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$ in $L^2(Q)^N$. This holds, for instance, for domains with Lipschitz boundaries, but also, since the sequence $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$ is uniformly bounded in $L^\infty(Q)^N$, for a larger class of bounded open subsets of $\mathbb{R}^{n+1}$. However, the approximation by semilinear parabolic equations yields immediately an additional regularity of these strong solutions.

Indeed, from the definitions of θ_ε and Θ_ε, from (18) with arbitrary $\varphi \in \mathcal{D}(Q)$, $\varphi \geq 0$, we find

$$(26)\quad f_i - \sum_{\substack{1\leq\, i_1<\ldots<\, i_k\leq\, N \\ i\,\in\,\{i_1,\ldots,i_k\}}} \frac{1}{k}\big(f_{i_1}+\cdots+f_{i_k}\big)^+ \leq P u_i^\varepsilon$$
$$= f_i - \Theta_\varepsilon(\boldsymbol{u}^\varepsilon) \leq f_i + f_i^- = f_i^+ \quad \text{a.e. in } Q.$$

By the conclusion of Theorem 2.3 we also obtain, for each $i = 1,\ldots,N$,

$$(27)\qquad f_i - \sum_{\substack{1\leq\, i_1<\ldots<\, i_k\leq\, N \\ i\,\in\,\{i_1,\ldots,i_k\}}} \frac{1}{k}(f_{i_1}+\cdots+f_{i_k})^+ \leq P u_i \leq f_i^+ \qquad \text{a.e. in } Q$$

and we can apply directly the second order linear parabolic theory (see [19]) in the Sobolev spaces

$$W_p^{2,1}(Q) = W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)), \qquad 1 < p < \infty.$$

These spaces satisfy the Sobolev imbeddings, for $p > (n+2)/(2-k)$, with $k = 0, 1$,

$$W_p^{2,1}(Q) \subset C_\alpha^{k,0}(\overline{Q}), \qquad 0 \leq \alpha < 2 - k - (n+2)/p,$$

where $C_\alpha^{k,0}(\overline{Q})$ denotes the spaces of Hölder continuous functions v in Q, with exponent α in the x-variables and $\alpha/2$ in the t-variable and, in the case $k = 1$, with ∇v satisfying the same property (see [19], p. 80). Therefore, as a consequence of (27), we conclude

Theorem 2.4. *Assume that $\partial\Omega$ is smooth, say of class C^2 and*

$$(28)\qquad \boldsymbol{f} \in L^p(Q)^N \qquad \textit{and} \qquad \boldsymbol{u}_0 \in \mathbb{K} \cap W^{2-2/p,p}(\Omega)^N, \qquad 1 < p < \infty,$$

with each component u_{0i} satisfying the compatibility condition $\dfrac{\partial u_{0i}}{\partial \boldsymbol{n}} = 0$ on $\partial\Omega$ if $p > 3$.

Then the unique solution $\boldsymbol{u}$ of the variational inequality (6)-(7) *is such that*

$$(29)\qquad \boldsymbol{u} \in W_p^{2,1}(Q)^N \cap L^\infty(0,T;\mathbb{K}),$$

and, in particular, is Hölder continuous in $\overline{Q}$ if $p > (n+2)/2$ and has $\nabla\boldsymbol{u}$ also Hölder continuous if $p > n+2$. □

We observe that, when $p < 2$, the inclusion $W_p^{2,1}(Q) \subset L^2(0,T;H^1(\Omega))$ only takes place if $p \geq (2n+4)/(n+4)$ but, as we shall see in the next section and since $\mathbb{K}$ is bounded, (6)-(7) is solvable for any $\boldsymbol{f} \in L^1(Q)^N$.

3. Existence and uniqueness of variational solutions

In this section, requiring the compactness of the inclusion of $H^1(\Omega)$ into $L^2(\Omega)$ by assuming a Lipschitz boundary $\partial\Omega$, we show how we can still solve the variational inequality (25) for a more general initial condition

$$\boldsymbol{u}_0 \in \tilde{\mathbb{K}} = \{v \in L^2(\Omega)^N : \sum_{j=1}^N v_j \leq 1,\ v_i \geq 0,\ i = 1, \ldots, N,\ \text{in } \Omega\} \tag{30}$$

and for general nonlinear $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{u})$ defining a continuous operator from $L^2(0,T;\tilde{\mathbb{K}})$ in $L^1(Q)^N$. We shall assume that $\boldsymbol{f} = \boldsymbol{f}(x,t,\boldsymbol{v}) : Q \times [0,1]^N \to \mathbb{R}^N$ satisfies

$$\boldsymbol{f} = \boldsymbol{f}(x,t,\boldsymbol{v}) \text{ is continuous in } \boldsymbol{v} \text{ for a.e. } (x,t) \in Q, \tag{31}$$

$$\exists\, \varphi_1 \in L^1(Q) : \quad |\boldsymbol{f}(x,t,\boldsymbol{v})| \leq \varphi_1(x) \quad \forall\, \boldsymbol{v} \in [0,1]^N, \quad \text{for a.e. } (x,t) \in Q. \tag{32}$$

However, now the solution has less regularity, namely

$$\boldsymbol{u} \in C([0,T]; L^2(\Omega)^N \cap \tilde{\mathbb{K}}) \cap L^2(0,T;H^1(\Omega)^N) \tag{33}$$

and its derivative may not be a function, since we only have

$$\partial_t \boldsymbol{u} \in L^1(Q)^N + L^2\big(0,T;\big(H^1(\Omega)^N\big)'\big). \tag{34}$$

Hence the first term in the variational inequality (25)-(30) should be interpreted in the duality sense between $L^1(Q)^N + L^2\big(0,T;\big(H^1(\Omega)^N\big)'\big)$ and $L^\infty(Q)^N \cap L^2(0,T;H^1(\Omega)^N)$, namely through the formula

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle_t = \int_{Q_t} P\boldsymbol{u}\cdot\boldsymbol{v} - \int_{Q_t} \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}, \qquad \forall\, \boldsymbol{v} \in L^\infty(Q)^N \cap L^2(0,T;H^1(\Omega)^N), \tag{35}$$

for arbitrary $t \in (0,T]$ since, as we shall see, (27) yields $P\boldsymbol{u} \in L^1(Q)^N$.

Theorem 3.1. *Under the assumptions* (30)*,* (31) *and* (32)*, the variational inequality* (25) *has a solution* $\boldsymbol{u}$ *satisfying* (33)*,* (34)*,* (27) *and* $\boldsymbol{u}(0) = \boldsymbol{u}_0$ *and we can write*

$$\int_Q \big(P\boldsymbol{u} - \boldsymbol{f}(\boldsymbol{u})\big)\cdot(\boldsymbol{v} - \boldsymbol{u}) \geq 0, \qquad \forall\, \boldsymbol{v} \in L^2(0,T;\tilde{\mathbb{K}}). \tag{36}$$

Proof. We consider the closed convex subset of $L^2(Q)^N$

$$\boldsymbol{K} = L^2(0,T;\tilde{\mathbb{K}}) = \{\boldsymbol{v} \in L^2(Q)^N : u_i \geq 0,\, i = 1, \ldots, N,\ \sum_{i=1}^N u_i \leq 1 \text{ in } Q\}$$

and we define $\Phi : \boldsymbol{K} \to \boldsymbol{K}$ as the nonlinear operator that associates to each $\boldsymbol{w} \in \boldsymbol{K}$ the solution $\boldsymbol{u}_{\boldsymbol{w}} = \Phi(\boldsymbol{w})$ of the variational inequality (25) with $\boldsymbol{f}$ replaced by $\boldsymbol{g} = \boldsymbol{f}(x,t,\boldsymbol{w})$ and fixed initial data $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$.

By showing that Φ is a continuous and compact operator, a fixed point $\boldsymbol{u} = \Phi(\boldsymbol{u})$, given by Schauder Theorem, will provide a solution with the required properties.

Indeed, first we observe that if we consider any sequence $\boldsymbol{K} \ni \boldsymbol{w}_\nu \xrightarrow[\nu]{} \boldsymbol{w} \in \boldsymbol{K}$ in $L^2(Q)^N$, by (31) and (32), the Lebesgue Theorem implies

$$\boldsymbol{g}_\nu = \boldsymbol{f}(\boldsymbol{w}_\nu) \xrightarrow[\nu]{} \boldsymbol{f}(\boldsymbol{w}) = \boldsymbol{g} \qquad \text{in } L^1(Q)^N.$$

Next, for any $\boldsymbol{g} \in L^1(Q)^N$ and any $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$ we consider sequences $\boldsymbol{g}_\nu \in L^2(Q)^N$ and $\boldsymbol{u}_{0\nu} \in \mathbb{K}$ such that

$$\boldsymbol{g}_\nu \xrightarrow[\nu]{} \boldsymbol{g} \text{ in } L^1(Q)^N \qquad \text{and} \qquad \boldsymbol{u}_{0\nu} \xrightarrow[\nu]{} \boldsymbol{u}_0 \text{ in } L^2(\Omega)^N$$

and we denote by $\boldsymbol{u}_\nu \equiv S(\boldsymbol{u}_{0\nu}, \boldsymbol{g}_\nu)$ the unique solution of (25)-(7) given by Theorem 2.3, for each $\boldsymbol{g}_\nu$ and $\boldsymbol{u}_{0\nu}$. We observe that each component of $P\boldsymbol{u}_\nu$ satisfies the inequality (27) with f_i replaced by $(\boldsymbol{g}_\nu)_i$. From (25) for $\boldsymbol{u}_\mu$ and $\boldsymbol{u}_\nu$, we easily find, for a.e. $t \in (0,T)$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}_\mu - \boldsymbol{u}_\nu|^2 + \int_\Omega |\nabla(\boldsymbol{u}_\mu - \boldsymbol{u}_\nu)|^2 \leq \int_\Omega (\boldsymbol{g}_\mu - \boldsymbol{g}_\nu)\cdot(\boldsymbol{u}_\mu - \boldsymbol{u}_\nu)$$

and, integrating in time, we obtain

$$\sup_{0<t<T}\int_\Omega |\boldsymbol{u}_\mu(t) - \boldsymbol{u}_\nu(t)|^2 + \int_Q |\nabla(\boldsymbol{u}_\mu - \boldsymbol{u}_\nu)|^2 \leq \int_\Omega |\boldsymbol{u}_{0\mu} - \boldsymbol{u}_{0\nu}|^2 + 4\int_Q |\boldsymbol{g}_\mu - \boldsymbol{g}_\nu|. \tag{37}$$

This estimate shows that $\{\boldsymbol{u}_\nu\}_\nu$ is a Cauchy sequence in the Banach space

$$\boldsymbol{W} = C([0,T]; L^2(\Omega)^N) \cap L^2(0,T; H^1(\Omega)^N) \tag{38}$$

with respect to the norm

$$|||\boldsymbol{v}||| = \Big(\sup_{0<t<T}\int_\Omega |\boldsymbol{v}(t)|^2 + \int_Q |\nabla \boldsymbol{v}|^2\Big)^{1/2} \tag{39}$$

and, hence, there exists a function $\boldsymbol{u}_{\boldsymbol{g}} \in \boldsymbol{W}$

$$\boldsymbol{u}_\nu \xrightarrow[\nu]{} \boldsymbol{u}_{\boldsymbol{g}} \qquad \text{in } \boldsymbol{W}.$$

In addition, $\boldsymbol{u}_{\boldsymbol{g}} \in L^2(0,T;\mathbb{K}) \cap C([0,T];\tilde{\mathbb{K}})$ and, recalling (27), also $P\boldsymbol{u}_{\boldsymbol{g}} \in L^1(Q)^N$, which implies, by (35), that $\partial_t \boldsymbol{u}_{\boldsymbol{g}}$ satisfies (34). Hence, using (35), we may pass to the limit in ν in

$$\langle P\boldsymbol{u}_\nu - \boldsymbol{g}_\nu, \boldsymbol{v} - \boldsymbol{u}_\nu\rangle = \int_Q (P\boldsymbol{u}_\nu - \boldsymbol{g}_\nu)\cdot(\boldsymbol{v} - \boldsymbol{u}_\nu) \geq 0$$

for an arbitrary $\boldsymbol{v} \in L^2(0,T;\mathbb{K}) \subset L^\infty(Q)^N$, and using the formula

$$2\langle \partial_t \boldsymbol{u}_{\boldsymbol{g}}, \boldsymbol{u}_{\boldsymbol{g}}\rangle_t = \int_\Omega |\boldsymbol{u}_{\boldsymbol{g}}(t)|^2 - \int_\Omega |\boldsymbol{u}_0|^2, \qquad \forall\, t \in (0,T],$$

we conclude that $\boldsymbol{u}_{\boldsymbol{g}} = S(\boldsymbol{u}_0, \boldsymbol{g})$ is the (unique) solution of the variational inequality (25) (or equivalently (36)) with data $\boldsymbol{g} \in L^1(Q)^N$ and $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$. In particular, from (37), we also obtain that, for fixed $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$, the operator $\Sigma : \boldsymbol{g} \mapsto \boldsymbol{u}_{\boldsymbol{g}} = S(\boldsymbol{u}_0, \boldsymbol{g})$ is Hölder continuous of order $1/2$, from $L^1(Q)^N$ into $\boldsymbol{W}$.

Since $\partial_t \boldsymbol{u}_{\boldsymbol{g}}$ satisfies the property (34), it is in fact in $L^1(0,T;H^{-s}(\Omega)^N)$, for s sufficiently large and, by a well known compactness embedding (see [25] or Theorem 3.11 of [26]), the compactness of $H^1(\Omega) \subset L^2(\Omega)$ implies that, in fact, Σ regarded as an operator from $L^1(Q)^N$ into $\boldsymbol{K} \subset L^2(Q)^N$ is, therefore, completely continuous. Hence, $\Phi = \Sigma \circ \boldsymbol{f}$ fulfils the requirements of the Schauder fixed point theorem and the proof is complete. □

Remark 3.2. *It is clear that if $\boldsymbol{u}_0 \in \mathbb{K}$ and, in* (32), *$\varphi_1 \in L^2(Q)$, we obtain in* Theorem 3.1 *the existence of a strong solution satisfying* (23) *and* (24). *Of course, if we have the regularity assumptions of* Theorem 2.4, *i.e., $\varphi_1 \in L^p(Q)$, implying by the inequalities* (27) *that $P\boldsymbol{u} \in L^p(Q)^N$, we also obtain solutions in $W^{2,1}_p(Q)^N$, in particular Hölder continuous solutions if $p > (n+2)/2$.*

In general the problem (36)-(30) may have more than one solution, but if we assume, in addition, that for some $\lambda > 0$, $\lambda I - \boldsymbol{f}$ is monotone non-decreasing in $[0,1]^N$, i.e.

$$\begin{aligned}(40)\quad \exists\,\lambda > 0: \quad & \lambda|\boldsymbol{v}-\boldsymbol{w}|^2 \\ & - \big(\boldsymbol{f}(x,t,\boldsymbol{v}) - \boldsymbol{f}(x,t,\boldsymbol{w})\big)\cdot(\boldsymbol{v}-\boldsymbol{w}) \geq 0, \quad (x,t)\in Q,\ \forall\,\boldsymbol{v},\boldsymbol{w}\in[0,1]^N,\end{aligned}$$

in particular, if $\boldsymbol{f}$ is Lipschitz continuous in $\boldsymbol{v}$, then there exists at most one solution $\boldsymbol{u}$ of the variational inequality (25) in the class (33) and initial condition $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$.

In order to prove the uniqueness of solution, we suppose that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two solutions of the variational inequality (25) with initial condition $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$ and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{u}_1)$, $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{u}_2)$ respectively. Then, choosing $\boldsymbol{u}_2$ and $\boldsymbol{u}_1$ as test functions, respectively, using (40) we find

$$\begin{aligned}\frac{1}{2}\int_\Omega |\boldsymbol{u}_2(t)-\boldsymbol{u}_1(t)|^2 &+ \int_{Q_t} |\nabla(\boldsymbol{u}_2-\boldsymbol{u}_1)|^2 \\ &\leq \int_{Q_t} \big(\boldsymbol{f}(\boldsymbol{u}_2)-\boldsymbol{f}(\boldsymbol{u}_1)\big)\cdot\big(\boldsymbol{u}_2-\boldsymbol{u}_1\big) \leq \lambda\int_{Q_t}|\boldsymbol{u}_2-\boldsymbol{u}_1|^2\end{aligned}$$

and so, by Grownvall inequality $\boldsymbol{u}_1 = \boldsymbol{u}_2$ a.e. in Q, since $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$.

We redefine the variational formulation of the approximating problem (18) in the framework of this section with Θ_ε defined in (16) and with initial condition only in $L^2(\Omega)^N$,

$$\begin{aligned}(41)\quad \int_Q \partial_t\boldsymbol{u}^\varepsilon\cdot\boldsymbol{v} + \int_Q \nabla\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{v} + \int_Q \Theta_\varepsilon(\boldsymbol{u}^\varepsilon)\cdot\boldsymbol{v} &= \int_Q \boldsymbol{f}(\boldsymbol{u}^\varepsilon)\cdot\boldsymbol{v}, \\ \forall\boldsymbol{v}\in\ L^2(0,T;H^1(\Omega)^N)&\cap L^\infty(Q)^N.\end{aligned}$$

Arguing as in Theorem 3.1 we may prove the existence of a solution of the approximating problem (12), with initial condition $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$ as long as $\boldsymbol{f}$ satisfies (31) and (32). We also have uniqueness if we assume (40).

Theorem 3.3. *Suppose that $\boldsymbol{f}$ satisfies* (31), (32) *and* (40) *and $\boldsymbol{u}_0 \in \tilde{\mathbb{K}}$.*

Let $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{u}$ be, respectively, the unique solution of the approximating problem (12) *and of the variational inequality* (25)*, both with initial condition $\boldsymbol{u}_0$. Then there exists a positive constant $c = c(\varphi_1, T)$ such that the following estimate in the norm* (39) *of $\boldsymbol{W} = C([0,T];L^2(\Omega)^N)\cap L^2(0,T;H^1(\Omega)^N)$ holds,*

$$(42)\qquad |||\boldsymbol{u}^\varepsilon - \boldsymbol{u}||| \leq c\sqrt{\varepsilon}.$$

Proof. We choose in (41) $\boldsymbol{v} = \boldsymbol{u}^\varepsilon - \boldsymbol{u}$ as test function. Since $\boldsymbol{u}\in\boldsymbol{K}$, then

$$\int_Q \Theta_\varepsilon(\boldsymbol{u}^\varepsilon)\cdot(\boldsymbol{u}^\varepsilon-\boldsymbol{u}) \geq 0$$

and so

$$(43)\qquad \int_{Q_t}\partial_t\boldsymbol{u}^\varepsilon\cdot(\boldsymbol{u}^\varepsilon-\boldsymbol{u}) + \int_{Q_t}\nabla\boldsymbol{u}^\varepsilon\cdot(\boldsymbol{u}^\varepsilon-\boldsymbol{u}) \leq \int_{Q_t}\boldsymbol{f}(\boldsymbol{u}^\varepsilon)\cdot(\boldsymbol{u}^\varepsilon-\boldsymbol{u}).$$

Choosing as test function in (25) $\boldsymbol{v}^\varepsilon = \big((u_1^\varepsilon - \frac{\varepsilon}{N})^+, \dots, (u_N^\varepsilon - \frac{\varepsilon}{N})^+\big)$ we get

$$\int_{Q_t} \partial_t \boldsymbol{u} \cdot (\boldsymbol{u}^\varepsilon - \boldsymbol{u}) + \int_{Q_t} \nabla \boldsymbol{u} \cdot \nabla(\boldsymbol{u}^\varepsilon - \boldsymbol{u}) \geq \int_{Q_t} \boldsymbol{f}(\boldsymbol{u}) \cdot (\boldsymbol{u}^\varepsilon - \boldsymbol{u}) + \int_{Q_t} \big[P\boldsymbol{u} - \boldsymbol{f}(\boldsymbol{u})\big] \cdot (\boldsymbol{u}^\varepsilon - \boldsymbol{v}^\varepsilon) \tag{44}$$

and subtracting (44) from (43) we get

$$\begin{aligned} \frac{1}{2}\int_\Omega |\boldsymbol{u}^\varepsilon(t) - \boldsymbol{u}(t))|^2 + \int_{Q_t} |\nabla(\boldsymbol{u}^\varepsilon - \boldsymbol{u})|^2 &\leq \int_{Q_t} (\boldsymbol{f}(\boldsymbol{u}^\varepsilon) - \boldsymbol{f}(\boldsymbol{u})) \cdot (\boldsymbol{u}^\varepsilon - \boldsymbol{u}) + \int_{Q_t} \big[P\boldsymbol{u} - \boldsymbol{f}(\boldsymbol{u})\big] \cdot (\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon) \\ &\leq \lambda \int_{Q_t} |\boldsymbol{u}^\varepsilon - \boldsymbol{u}|^2 + \varepsilon \int_{Q_t} |P\boldsymbol{u} - \boldsymbol{f}(\boldsymbol{u})|, \end{aligned} \tag{45}$$

since $\|\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon\|_{L^\infty(Q)^N} \leq \varepsilon$. Letting $C = C(\varphi_1, T) = \|P\boldsymbol{u} - \boldsymbol{f}(\boldsymbol{u})\|_{L^1(Q)^N}$, from (45) we obtain first, by application of the Grownwall inequality,

$$\int_\Omega |\boldsymbol{u}^\varepsilon(t) - \boldsymbol{u}(t)|^2 \leq 2\varepsilon C e^{2\lambda t}$$

and using again (45), also

$$|||\boldsymbol{u}^\varepsilon - \boldsymbol{u}||| \leq c\sqrt{\varepsilon}.$$

□

With similar arguments we may give a continuous dependence result for solutions of the variational inequality (36).

Suppose we have a sequence $\boldsymbol{f}^\nu \xrightarrow[\nu]{} \boldsymbol{f}$ in the following sense

$$\left.\begin{aligned} &\boldsymbol{f}^\nu = \boldsymbol{f}^\nu(x,t,\boldsymbol{v}) \text{ are continuous in } \boldsymbol{v} \in [0,1]^N, \text{ for a.e. } (x,t) \in Q \\ &\boldsymbol{f}^\nu(\cdot,\cdot,\boldsymbol{v}) \xrightarrow[\nu]{} \boldsymbol{f}(\cdot,\cdot,\boldsymbol{v}) \text{ in } L^1(Q)^N \text{ for all fixed } \boldsymbol{v} \in [0,1]^N. \end{aligned}\right\} \tag{46}$$

In addition, the assumption (32) is satisfied for all $\boldsymbol{f}$ uniformly in ν, i.e., there is a common φ_1 such that (32) holds for all ν, and the initial data are such that

$$\tilde{\mathbb{K}} \ni \boldsymbol{u}_0^\nu \xrightarrow[\nu]{} \boldsymbol{u}_0 \qquad \text{in } L^2(\Omega)^N. \tag{47}$$

Hence, by Theorem 3.1, it is clear that there are solutions $\{\boldsymbol{u}^\nu\}_{\nu\in\mathbb{N}}$ to the corresponding problems associated with $\boldsymbol{f}^\nu$ and $\boldsymbol{u}_0^\nu$ and, moreover, they satisfy (33) and (34) uniformly in ν, i.e., their norms in those spaces are bounded by a constant independent of ν. Therefore, we have a function $\boldsymbol{u}$ in the same class (33) and (34), and a subsequence, still denoted by ν, such that

$$\boldsymbol{u}^\nu \xrightarrow[\nu]{} \boldsymbol{u} \text{ in } L^2(0,T;H^1(\Omega)^N) \text{ weak} \quad \text{and in} \quad L^\infty(0,T;\tilde{\mathbb{K}}) \text{ weak-}* \tag{48}$$

$$\boldsymbol{u}^\nu \xrightarrow[\nu]{} \boldsymbol{u} \quad \text{a.e. in } Q \quad \text{and in} \quad L^p(Q)^N, \quad \forall\, 1 \leq p < \infty. \tag{49}$$

By assumption (46) and Lebesgue Theorem, we conclude that $\boldsymbol{f}^\nu(\boldsymbol{u}^\nu) \xrightarrow[\nu]{} \boldsymbol{f}(\boldsymbol{u})$ a.e. in Q and in $L^1(Q)^N$, as well as

$$\int_Q \boldsymbol{f}^\nu(\boldsymbol{u}^\nu) \cdot \boldsymbol{u}^\nu \xrightarrow[\nu]{} \int_Q \boldsymbol{f}(\boldsymbol{u}) \cdot \boldsymbol{u}, \tag{50}$$

$$\int_Q \big(\boldsymbol{f}^\nu(\boldsymbol{u}^\nu) - \boldsymbol{f}(\boldsymbol{u})\big) \cdot (\boldsymbol{u}^\nu - \boldsymbol{u}) \xrightarrow[\nu]{} 0, \tag{51}$$

since, in particular, $|\boldsymbol{u}^\nu| \leq 1$ and $|\boldsymbol{u}| \leq 1$ a.e. in Q.

Recalling (27) for each ν, we may take the limit in

$$\int_Q \big(P\boldsymbol{u}^\nu - \boldsymbol{f}^\nu(\boldsymbol{u}^\nu)\big) \cdot (\boldsymbol{v} - \boldsymbol{u}^\nu) \geq 0 \tag{52}$$

for a fixed $\boldsymbol{v} \in L^2(0,T;\tilde{\mathbb{K}})$. Using (50) and (48), that in particular imply, by standard arguments, already used in the proofs of Theorems 2.3 and 3.1,

$$P\boldsymbol{u} \in L^1(Q)^N \qquad \text{and} \qquad \liminf_\nu \int_{Q_t} P\boldsymbol{u}^\nu \cdot \boldsymbol{u}^\nu \geq \int_{Q_t} P\boldsymbol{u} \cdot \boldsymbol{u}, \quad \forall\, t \in (0,T),$$

we conclude that $\boldsymbol{u}$ is a solution of (36) with initial condition $\boldsymbol{u}_0$.

Using $\boldsymbol{v} = \boldsymbol{u}\chi_{(0,t)} + \boldsymbol{u}^\nu\chi_{(t,T)}$ in (52) and $\boldsymbol{v} = \boldsymbol{u}^\nu\chi_{(0,t)} + \boldsymbol{u}\chi_{(t,T)}$ in (36) we find, for a.e. $t \in (0,T)$,

$$\frac{1}{2}\int_\Omega |\boldsymbol{u}^\nu(t) - \boldsymbol{u}(t)|^2 + \int_{Q_t} |\nabla(\boldsymbol{u}^\nu - \boldsymbol{u})|^2 \leq \int_{Q_t} \big[\boldsymbol{f}^\nu(\boldsymbol{u}^\nu) - \boldsymbol{f}(\boldsymbol{u})\big] \cdot (\boldsymbol{u}^\nu - \boldsymbol{u}) + \frac{1}{2}\int_\Omega |\boldsymbol{u}_0^\nu - \boldsymbol{u}_0|^2$$

and, by (51), we conclude that $\boldsymbol{u}^\nu \xrightarrow[\nu]{} \boldsymbol{u}$ strongly in $\boldsymbol{W}$. Therefore, we have proved the following result

Theorem 3.4. *If $\boldsymbol{u}^\nu$ denotes the solution to the variational inequality* (36) *with $\boldsymbol{f}^\nu$ satisfying the assumptions* (46) *and* (32) *uniformly in ν and initial condition satisfying* (47)*, then there exists a subsequence $\{\boldsymbol{u}^\nu\}_{\nu\in\mathbb{N}}$ such that*

$$\boldsymbol{u}^\nu \xrightarrow[\nu]{} \boldsymbol{u} \ \text{ in } \ C([0,T]; L^2(\Omega)^N \cap \tilde{\mathbb{K}}) \cap L^2(0,T; H^1(\Omega)^N) \cap L^p(Q)^N, \ \forall\, 1 \leq p < \infty,$$

where $\boldsymbol{u}$ is a solution to (36) *corresponding to the limit $\boldsymbol{f}$ and the limit initial condition $\boldsymbol{u}_0$. In addition, if $\boldsymbol{f}$ satisfies* (40)*, by uniqueness of $\boldsymbol{u}$, the whole sequence $\{\boldsymbol{u}^\nu\}_{\nu\in\mathbb{N}}$ converges.* □

4. The multiphases system and its characterization

In this section we consider a variational solution $\boldsymbol{u}$ of (25) obtained in Theorem 3.1, i.e., satisfying (33) and (34). Setting

$$w_i(\boldsymbol{u}) = 1 - \sum_{j\neq i} u_j, \qquad i = 1, \ldots, N, \tag{53}$$

each component u_i satisfies a double obstacle problem

$$0 \leq u_i(x,t) \leq w_i(x,t) \qquad \text{a.e. } (x,t) \in Q, \qquad i = 1, \ldots, N. \tag{54}$$

For an arbitrary nonnegative and bounded function $\varphi = \varphi(x,t)$ defined for $(x,t) \in Q$, such that

$$\mathbb{K}_0^\varphi = \{v \in L^2(0,T;H^1(\Omega)) : \ 0 \leq v \leq \varphi \ \text{ in } Q\} \neq \emptyset, \tag{55}$$

and for a given $g \in L^1(Q)$, we may introduce the parabolic double obstacle scalar problem

$$(56)\quad u \in \mathbb{K}_0^\varphi : \qquad \int_Q \partial_t u(v-u) + \int_Q \nabla u \cdot \nabla(v-u) \geq \int_Q g(v-u) \quad \forall v \in \mathbb{K}_0^\varphi,$$

subject to a given compatible initial condition

$$(57)\qquad u(0) = u_0 \qquad \text{in } \Omega.$$

For each $i = 1, \ldots, N$, we have $u_i \in \mathbb{K}_0^{w_i}$ and, by choosing in (25) $\boldsymbol{v} \in L^2(0,T;\mathbb{K})$, such that $v_j = u_j$ for $j \neq i$ and $v_i = v \in \mathbb{K}_0^{w_i}$ arbitrarily, it is clear that u_i is a solution of the scalar double obstacle problem (56) with $\varphi = w_i$ and $g = f_i(\boldsymbol{u})$. Hence we can obtain further properties of our solution by applying the general theory of the obstacle problem. For the sake of completeness we prove here the result below.

Let

$$(58)\qquad \varphi \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q) \ \text{ with } \varphi \geq 0 \text{ a.e. in } Q,$$

$$(59)\quad \partial_t \varphi \in L^2\big(0,T;\big(H^1(\Omega)\big)'\big) \ \text{ with } P\varphi \in L^1(Q), \qquad \frac{\partial \varphi}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega \times (0,T),$$

and

$$(60)\qquad g \in L^1(Q), \qquad u_0 \in L^2(\Omega), \quad 0 \leq u_0 \leq \varphi(0) \ \text{ in } \Omega.$$

We observe that (59) means that φ satisfies the formula

$$\langle \partial_t \varphi, v \rangle_t = \int_{Q_t} v\, P\varphi - \int_{Q_t} \nabla\varphi \cdot \nabla v, \qquad \forall v \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q).$$

Proposition 4.1. *Under the assumptions* (58)-(60) *the unique solution* $u \in \mathbb{K}_0^\varphi$ *to the scalar problem* (56)-(57) *is such that*

$$(61)\quad u \in C([0,T];L^2(\Omega)) \cap L^\infty(Q), \qquad \partial_t u \in L^1(Q) + L^2(0,T;\big(H^1(\Omega)'\big),$$

and it satisfies the parabolic semilinear equation

$$(62)\qquad Pu = g + g^- \chi_{\{u=0\}} - (P\varphi - g)^- \chi_{\{u=\varphi\}} \qquad \text{a.e. in } Q.$$

Proof. Using the function θ_ε given by (11) and defining

$$(63)\qquad h_\varepsilon(v) = g^- \theta_\varepsilon(v) - (P\varphi - g)^- \theta_\varepsilon(\varphi - v)$$

we can consider the approximating problem, for $\varepsilon > 0$,

$$(64)\qquad \int_Q \big(Pu^\varepsilon + h_\varepsilon(u^\varepsilon)\big)v = \int_Q gv, \qquad \forall\, v \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q),$$

with the initial condition $u^\varepsilon(0) = u_0$ in Ω. Since h_ε is monotone and φ is bounded, arguing as in Theorem 3.1, the problem (64) has a unique solution u^ε in the class (61). Moreover, it satisfies

$$(65)\qquad -\varepsilon \leq u^\varepsilon \leq \varphi + \varepsilon \qquad \text{a.e. in } Q,$$

as we can show by choosing, in (64), $v = (-u^\varepsilon - \varepsilon)^+$ and $v = (u^\varepsilon - \varphi - \varepsilon)^+$, respectively. Indeed, in the first case we have

$$\int_Q vPv = -\int_Q vPu^\varepsilon = \int_{\{v>0\}} v\,(h(u^\varepsilon) - g) = \int_{\{u^\varepsilon < -\varepsilon\}} (-g^- - g) \leq 0,$$

since $h_\varepsilon(u^\varepsilon) = -1$ and $h_\varepsilon(\varphi - u_\varepsilon) = 0$, because $u^\varepsilon < -\varepsilon$ and $\varphi - u_\varepsilon > \varepsilon$, and, in the second case,

$$\int_Q vPv = \int_Q vP(u^\varepsilon - \varphi) = \int_{\{v>0\}} v\,(g - h(u^\varepsilon) - P\varphi)$$
$$= \int_{\{\varphi - u^\varepsilon > \varepsilon\}} \big(-(P\varphi - g) - (P\varphi - g)^-\big) \le 0,$$

since $h_\varepsilon(\varphi - u^\varepsilon) = -1$ and $h_\varepsilon(u^\varepsilon) = 0$ if $\varphi - u_\varepsilon < -\varepsilon$ and $u^\varepsilon > \varphi + \varepsilon$.

Hence, using the monotonicity argument, we easily conclude that $u = \lim\limits_{\varepsilon\to 0} u^\varepsilon \in \mathbb{K}_0^\varphi$ is the unique solution of the variational inequality (56). Remarking that, from (63) we have

$$-g^- \le h_\varepsilon(u^\varepsilon) \le (P\varphi - g)^- \qquad \text{a.e. in } Q,$$

from (64) we deduce in the limit the Lewy-Stampacchia inequalities

$$(P\varphi - g)^- \le Pu - g \le g^- \qquad \text{a.e. in } Q.$$

In particular, this yields $Pu \in L^1(Q)$ and (56) implies that u also solves

$$\int_Q \big(Pu - g\big)(v - u) \ge 0, \qquad \forall\, v \in \tilde{\mathbb{K}}_0^\varphi, \tag{66}$$

where $\tilde{\mathbb{K}}_0^\varphi = \{v \in L^2(Q) : 0 \le v \le \varphi \text{ in } Q\} \subset L^\infty(Q)$.

Let $\mathcal{O} \subset Q$ be an arbitrary measurable set and set $v = u$ in $Q \setminus \mathcal{O}$ and $v = \delta\varphi$ in $\mathcal{O}$, with $\delta \in [0,1]$, in (66). Since $\mathcal{O}$ is arbitrary, we conclude the pointwise inequality

$$\big(Pu - g\big)(\phi - u) \ge 0 \qquad \forall\, \phi \in [0, \varphi(x,t)] \ \text{ a.e. in } Q, \tag{67}$$

which implies, up to null measure subsets of Q,

$$Pu - g \ge 0 \ \text{ in } \ \{u = 0\}, \qquad Pu - g \le 0 \ \text{ in } \ \{u = \varphi\}, \tag{68}$$

$$Pu = g \ \text{ in } \ \Lambda = \{0 < u < \varphi\}. \tag{69}$$

On the other hand, arguing as in Lemma 2 of [24] and noting that $V = (u, -\nabla u) \in L^1(Q)^{n+1}$ and $D \cdot V = Pu \in L^1(Q)$, with $D = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$, we have

$$Pu = 0 \ \text{ a.e. in } \ \{u = 0\} \qquad \text{and} \qquad Pu = P\varphi \ \text{ a.e. in } \ \{u = \varphi\}.$$

Hence, by (68), up to negligible sets, we have $\{u = 0\} \subset \{g \le 0\}$ and $\{u = \varphi\} \subset \{P\varphi \le g\}$, and using also (69), we finally conclude (62). □

Theorem 4.2. *Any solutions $\boldsymbol{u}$ of the variational inequality* (25) (*or* (36)) *under the conditions of Theorem* 3.1 *satisfy the semilinear parabolic system*

$$\begin{aligned} Pu_i = {} & f_i(\boldsymbol{u}) + f_i^-(\boldsymbol{u})\chi_{\{u_i=0\}} \\ & - \sum_{\substack{1\le\, i_1 < \dots < \, i_k \le\, N \\ i\, \in\, \{i_1, \dots, i_k\}}} \frac{1}{k}\big(f_{i_1}(\boldsymbol{u}) + \dots + f_{i_k}(\boldsymbol{u})\big)^+ \chi_{i_1\dots i_k} \qquad \text{a.e. in } Q, \end{aligned} \tag{70}$$

where $\chi_{i_1\dots i_k} = \chi_{I_{i_1\dots i_k}}$, for $k = 1, \dots, N$, denotes the characteristic function of

$$I_{i_1\dots i_k} = \{(x,t) \in Q : \ u_{i_1\dots i_k}(x,t) = 1,\ u_{i_j}(x,t) > 0 \text{ for all } j = 1, \dots, k\}. \tag{71}$$

Proof. We notice that the regularity (58), (59) holds for $w_i = 1 - \sum_{j\neq i} u_j$, so w_i can be chosen as the upper obstacle of each component u_i, $i = 1, \dots, N$, of $\boldsymbol{u}$, to which we can apply the conclusions of Proposition 4.1. Since $\{u_i = 0\} \subset \{f_i(\boldsymbol{u}) \leq 0\}$ a.e., for each $i = 1, \dots, N$, we have

$$Pu_i = f_i(\boldsymbol{u}) + f_i^-(\boldsymbol{u})\chi_{\{u_i=0\}} - (Pw_i - f_i(\boldsymbol{u}))^-\chi_{\{u_i=w_i,\, u_i>0\}} \qquad \text{in } Q, \tag{72}$$

and the condition (70) will follow if we show that

$$(Pw_i - f_i(\boldsymbol{u}))^-\chi_{\{u_i=w_i,\, u_i>0\}} = \sum_{\substack{1\leq\, i_1<\dots<\, i_k\leq\, N \\ i\,\in\,\{i_1,\dots,i_k\}}} \frac{1}{k}\big(f_{i_1}(\boldsymbol{u}) + \dots + f_{i_k}(\boldsymbol{u})\big)^+\chi_{i_1\dots i_k} \ \text{ in } Q, \tag{73}$$

Observe that

$$\{u_i = w_i,\, u_i > 0\} = \bigcup_{\substack{1\leq\, i_1<\dots<\, i_k\leq\, N \\ i\,\in\,\{i_1,\dots,i_k\}}} I_{i_1\dots i_k},$$

and these sets are a.e. disjoint. Here the union is taken also over all the subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, N\}$ that include i and over all $k = 1, \dots, N$. We remark that $Pw_i = Pu_i$ in that subset and

- in the sets $I_i = \{u_i = 1\}$, $Pw_i = 0$ and $(Pw_i - f_i(\boldsymbol{u}))^- = f_i(\boldsymbol{u})^+$, for $i = 1, \dots, N$;
- in each set $I_{i_1\dots i_k}$, for $k \geq 2$, as we shall see,
$$(Pu_i - f_i(\boldsymbol{u}))^- = \frac{1}{k}\big(f_{i_1}(\boldsymbol{u}) + \dots + f_{i_k}(\boldsymbol{u})\big)^+,$$
and this fact concludes the proof.

Let $(x_0, t_0) \in I_{i_1\dots i_k}$. Recall that $\{i_1, \dots, i_k\}$ is the set of indexes for which we have $0 < u_{i_j}(x_0, t_0)$ (notice that $i \in \{i_1, \dots, i_k\}$). Denoting $\alpha = \min\{u_{i_j}(x_0, t_0) : j = 1, \dots, k\}$, the set $\mathcal{O} = \bigcap_{j=1}^k \{u_{i_j} > \alpha/2\}$ is measurable and contains (x_0, t_0). Given any measurable set $\omega \subset \mathcal{O}$, choose, in (36), as test function $\boldsymbol{v} = (v_1, \dots, v_N)$ defined by

$$v_{i_1} = u_{i_1} \pm \delta\chi_\omega, \quad v_{i_j} = u_{i_j} \mp \delta\chi_\omega \text{ for a fixed } j \in \{2, \dots, k\}, \quad v_l = u_l \ \forall\, l \neq i_1, i_j,$$

observing that

$$\sum_{j=1}^N v_j = \sum_{j=1}^N u_j \pm \delta\chi_\omega \mp \delta\chi_\omega = \sum_{j=1}^N u_j \leq 1$$

and

$$v_j \geq 0,\ j = 1, \dots, N, \qquad \text{as long as } 0 < \delta \leq \alpha/2.$$

Returning to the inequality (36) and setting $S_j = Pu_j - f_j(\boldsymbol{u})$, we get

$$\pm\delta \int_Q S_{i_1}\chi_\omega \mp \delta \int_Q S_{i_j}\chi_\omega \geq 0.$$

Since $\omega \supset \{(x_0, t_0)\}$ was taken arbitrarily in $\mathcal{O}$ and (x_0, t_0) is a generic point of $I_{i_1\dots i_k}$, we conclude that

$$S_{i_1} = S_{i_j}, \qquad \text{a.e. in } I_{i_1\dots i_k}, \qquad \text{for any } j \in \{2, \dots, k\}. \tag{74}$$

Recalling that $\displaystyle\sum_{j=1}^{N} Pu_j = Pu_{i_1\dots i_k} = 0$, in the set $I_{i_1\dots i_k}$ we get, using (74), that

$$kS_{i_1} = S_{i_1} + \cdots + S_{i_k} = \big(Pu_{i_1} - f_{i_1}\big) + \cdots + \big(Pu_{i_k} - f_{i_k}\big) \\ = Pu_1 + \cdots + Pu_N - (f_{i_1} + \cdots + f_{i_k}),$$

where, for simplicity, we set $f_j = f_j(\boldsymbol{u})$, and so

$$S_i = S_{i_1} = -\frac{1}{k}\big(f_{i_1} + \cdots + f_{i_k}\big).$$

But in $I_{i_1\dots i_k}$ we have $S_i \leq 0$ (recall that $u_i = w_i$ and (68)) and so

$$(Pu_i - f_i(\boldsymbol{u}))^- = -(Pu_i - f_i(\boldsymbol{u})) = -S_i = \frac{1}{k}\big(f_{i_1} + \cdots + f_{i_k}\big) = \frac{1}{k}\big(f_{i_1} + \cdots + f_{i_k}\big)^+.$$

□

Corollary 4.3. *Let $\boldsymbol{u}$ be the solution of the variational inequality* (25) (*or* (36)) *under the conditions of Theorem* 3.1.

Then, denoting by $|A|$ the $(n+1)$-Lebesgue measure of $A \subset Q$, we have

$$\Big|\Big\{ \sum_{j=1}^{k} f_{i_j}(\boldsymbol{u}) < 0\Big\} \cap \Big\{ \sum_{j=1}^{k} u_{i_j} = 1,\, u_{i_j} > 0,\, j = 1, \dots, k\Big\}\Big| = 0 \tag{75}$$

for each partial coincidence subset $I_{i_1\dots i_k}$, as well as

$$\big|\{f_i(\boldsymbol{u}) > 0\} \cap \{u_i = 0\}\big| = 0, \qquad i = 1, \dots, N. \tag{76}$$

Proof. Being $I_{i_1\dots i_k}$ defined in (71), using the equation (70), we obtain, for each i_j with $j = 1, \dots, k$, denoting $f_{i_j} = f_{i_j}(\boldsymbol{u})$,

$$Pu_{i_j} = f_{i_j} - \frac{1}{k}\big(f_{i_1} + \cdots + f_{i_k}\big)^+ \qquad \text{a.e in } I_{i_1\dots i_k}.$$

Summing these k equations, we have

$$0 = \sum_{j=1}^{k} Pu_{i_j} = f_{i_1} + \cdots + f_{i_k} - \big(f_{i_1} + \cdots + f_{i_k}\big)^+ = \big(f_{i_1} + \cdots + f_{i_k}\big)^- \qquad \text{a.e in } I_{i_1\dots i_k}.$$

So, in $I_{i_1\dots i_k} = \big\{\sum_{j=1}^{k} u_{i_j} = 1,\, u_{i_j} > 0,\, j = 1, \dots, k\big\}$ we have $\displaystyle\sum_{j=1}^{k} f_{i_j} \geq 0$ a.e. and (75) follows.

The proof of (76) is similar (recall (68)). □

As a consequence of this corollary the semilinear system (70) can, in fact, be written in the equivalent form for $i = 1, \dots, N$,

$$Pu_i = f_i(\boldsymbol{u}) - f_i(\boldsymbol{u})\chi_{\{u_i=0\}} \\ - \sum_{\substack{1\leq i_1 < \dots < i_k \leq N \\ i \in \{i_1, \dots, i_k\}}} \frac{1}{k}(f_{i_1}(\boldsymbol{u}) + \cdots + f_{i_k}(\boldsymbol{u}))\chi_{i_1\dots i_k} \qquad \text{a.e. in } Q, \tag{77}$$

since $\{u_i = 0\} \subset \{f_i(\boldsymbol{u}) \leq 0\}$ and $I_{i_1\dots i_k} \subset \Big\{\displaystyle\sum_{j=1}^{k} f_{i_j}(\boldsymbol{u}) \geq 0\Big\}$ up to a negligible subset of Q.

This remark combined with the continuous dependence of the variational solutions obtained in Theorem 3.4 yields an interesting criteria of local stability of the characteristic functions of the coincidence sets in the Lebesgue measure. Denote

$$\chi^\nu_{i_1\dots i_k}=\chi_{\{u^\nu_{i_1\dots i_k}=1,\ u^\nu_{i_j}>0\ \forall j=1,\dots,k\}},\qquad 1\le i_1<\dots<i_k\le N,\quad k=1,\dots,N.$$

Theorem 4.4. *Let the assumptions and notations of* Theorem 3.4 *hold. Suppose that in some subset of positive measure $\omega\subseteq Q$ the following assumption on the limit problem holds*

$$\sum_{j=1}^{k} f_{i_j}(\boldsymbol{u})\neq 0 \qquad \text{a.e. in } \omega,\qquad 1\le i_1<\dots<i_k\le N,\quad k=1,\dots,N. \tag{78}$$

Then the associated characteristic functions are such that

$$\chi_{\{u^\nu_i=0\}}\xrightarrow[\nu]{}\chi_{\{u_i=0\}}\qquad \text{in } L^p(\omega),\qquad \forall\, i=1,\dots,N, \tag{79}$$

$$\chi^\nu_{i_1\dots i_k}\xrightarrow[\nu]{}\chi_{i_1\dots i_k}\qquad \text{in } L^p(\omega),\qquad \forall\, i_1,\dots,i_k, \tag{80}$$

for all p, $1<p<\infty$.

Proof. We observe that each $\boldsymbol{u}^\nu$ solves the system

$$Pu^\nu_i=f^\nu_i-f^\nu_i\chi_{\{u^\nu_i=0\}}-\sum_{\substack{1\le i_1<\dots<i_k\le N\\ i\in\{i_1,\dots,i_k\}}}\frac{1}{k}(f^\nu_{i_1}+\dots+f^\nu_{i_k})\,\chi^\nu_{i_1\dots i_k}\qquad \text{a.e. in } Q \tag{81}$$

where, for simplicity, we set $f^\nu_i=f^\nu_i(\boldsymbol{u}^\nu)$. By the convergence $\boldsymbol{u}^\nu\xrightarrow[\nu]{}\boldsymbol{u}$, we have $P\boldsymbol{u}^\nu\xrightarrow[\nu]{}P\boldsymbol{u}$ in the distributional sense. Since $0\le\chi^\nu_{i_1\dots i_k}\le 1$, there exists $\chi^*_{i_1\dots i_k}$, with $0\le\chi^*_{i_1\dots i_k}\le 1$ in Q, such that

$$\chi^\nu_{i_1\dots i_k}\xrightharpoonup[\nu]{}\chi^*_{i_1\dots i_k}\qquad \text{in}\quad L^\infty(Q)\text{ weak-}*\,.$$

Analogously, for some $\chi^*_{i,0}$, with $0\le\chi^*_{i,0}\le 1$ in Q, we have

$$\chi_{\{u^\nu_i=0\}}\xrightharpoonup[\nu]{}\chi^*_{i,0}\qquad \text{in}\quad L^\infty(Q)\text{ weak-}*\,.$$

We are going to prove that, in fact,

$$\chi^*_{i,0}=\chi_{\{u_i=0\}}\qquad\text{and}\qquad \chi^*_{i_1\dots i_k}=\chi_{i_1\dots i_k}\quad \text{a.e. in } \omega,$$

which concludes the proof, since the weak convergence to characteristic functions in $L^p(\omega)$ is in fact strong, as it is well known.

Passing to the limit in (81), we obtain

$$Pu_i=f_i-f_i\chi^*_{i,0}-\sum_{\substack{1\le i_1<\dots<i_k\le N\\ i\in\{i_1,\dots,i_k\}}}\frac{1}{k}(f_{i_1}+\dots+f_{i_k})\,\chi^*_{i_1\dots i_k}\qquad \text{a.e. in } Q$$

where, for simplicity, we have also set $f_{i_j}=f_{i_j}(\boldsymbol{u})$.

But each u_i also solves the equation (77), so, by subtraction, we obtain a.e. in Q,

$$-f_i\big(\chi_{\{u_i=0\}}-\chi^*_{i,0}\big)-\sum_{\substack{1\le i_1<\dots<i_k\le N\\ i\in\{i_1,\dots,i_k\}}}\frac{1}{k}(f_{i_1}+\dots+f_{i_k})\big(\chi_{i_1\dots i_k}-\chi^*_{i_1\dots i_k}\big)=0. \tag{82}$$

Noticing that $\chi_{\{u_i^\nu=0\}}u_i^\nu = 0$, passing to the limit, we get $\chi^*_{i,0}u_i = 0$, which means that $\chi^*_{i,0} = 0$ whenever $u_i > 0$. To conclude that $\chi^*_{i,0} = \chi_{\{u_i=0\}}$ we only need to prove that $\chi^*_{i,0} = 1$ if $u_i = 0$.

Recall that the sets $\{u_i = 0\}$ and $I_{i_1\ldots i_k}$, $1 \le i_1 < \ldots < i_k \le N$, $i \in \{i_1, \ldots, i_k\}$, $k = 1, \ldots, N$, are mutually disjoint. Hence in $\{u_i = 0\}$ we obtain

$$-f_i(1-\chi^*_{i,0}) + \sum_{\substack{1\le i_1<\ldots<i_k\le N\\ i\in\{i_1,\ldots,i_k\}}} \frac{1}{k}(f_{i_1}+\cdots+f_{i_k})\chi^*_{i_1\ldots i_k} = 0$$

and since the left hand side is nonnegative, by the assumption (78) we conclude that

$$\chi^*_{i,0} = 1 \qquad \text{and} \qquad \chi^*_{i_1\ldots i_k} = 0 \qquad \text{in } \{u_i = 0\}\cap\omega.$$

Since $\chi^\nu_{i_1\ldots i_k}(1-u^\nu_{i_1\ldots i_k}) = 0$ a.e. in Q, taking the limit in ν, we also obtain $\chi^*_{i_1\ldots i_k}(1-u_{i_1\ldots i_k}) = 0$ a.e in Q, i.e. $\chi^*_{i_1\ldots i_k} = 0$ if $u_{i_1\ldots i_k} < 1$. It remains to evaluate $\chi^*_{i_1\ldots i_k}$ when $u_{i_1\ldots i_k} = 1$ and $u_{i_j} > 0$, for all $j = 1, \ldots, k$ or when $u_{i_j} = 0$, for some $j = 1, \ldots, k$.

In this later case, where $u_{i_j} = 0$, for some $j = 1, \ldots, k$, we have $\chi_{i_1\ldots i_k} = 0$ and, since we already know that $\chi_{\{u_{i_j}=0\}} = \chi^*_{i_j,0}$, from (82) for the index i_j, we get

$$\sum_{\substack{1\le i_1<\ldots<i_k\le N\\ i_j\in\{i_1,\ldots,i_k\}}} \frac{1}{k}(f_{i_1}+\cdots+f_{i_k})\chi^*_{i_1\ldots i_k} = 0.$$

Then, by the assumption (78) we have $\chi^*_{i_1\ldots i_k} = 0$ in $\big(Q \setminus I_{i_1\ldots i_k}\big)\cap\omega$.

Finally, in $I_{i_1\ldots i_k}\cap\omega$, again from (82), we obtain

$$\frac{1}{k}(f_{i_1}+\cdots+f_{i_k})\big(1-\chi^*_{i_1\ldots i_k}\big) = 0$$

and the assumption (78) yields that $\chi^*_{i_1\ldots i_k} = 1$, completing the proof. □

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E-mail address: `rodrigue@ptmat.fc.ul.pt`
E-mail address: `lisa@math.uminho.pt`