# A CLASS OF STATIONARY NONLINEAR MAXWELL SYSTEMS

FERNANDO MIRANDA
*Department of Mathematics/CMat, University of Minho,*
*Campus de Gualtar, 4710-057 Braga, Portugal*
*fmiranda@math.uminho.pt*

JOSÉ-FRANCISCO RODRIGUES
*CMAF/FCUL, University of Lisbon,*
*Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal*
*rodrigue@fc.ul.pt*

LISA SANTOS
*Department of Mathematics/CMat, University of Minho,*
*Campus de Gualtar, 4710-057 Braga, Portugal*
*lisa@math.uminho.pt*

Preprint of an article submitted for consideration in
Mathematical Models and Methods in Applied Sciences (M3AS)

http://www.worldscinet.com/m3as

We study a new class of electromagnetostatic problems in the variational framework of the subspace of $\boldsymbol{W}^{1,p}(\Omega)$ of vector functions with zero divergence and zero normal trace, for $p > \frac{6}{5}$, in smooth, bounded and simply connected domains $\Omega$ of $\mathbb{R}^3$. We prove a Poincar-Friedrichs type inequality and we obtain the existence of steady-state solutions for an electromagnetic induction heating problem and for a quasi-variational inequality modelling a critical state generalized problem for type-II superconductors.

## 1. Introduction

Consider a nonlinear electromagnetic field in equilibrium in a bounded domain $\Omega$ of $\mathbb{R}^3$. The electromagnetic and the magnetic fields, respectively $\boldsymbol{e}$ and $\boldsymbol{h}$, satisfy the stationary generalized Maxwell's equations

$$\boldsymbol{j} = \nabla\times\boldsymbol{h}, \quad \nabla\times\boldsymbol{e} = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{h} = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $\boldsymbol{j}$ denotes the total current density and $\boldsymbol{f}$ denotes an internal magnetic current. The magnetic field $\boldsymbol{h}$ is supposed to be divergence free by assuming the usual relation with the magnetic induction field, $\boldsymbol{b} = \mu\,\boldsymbol{h}$, where the magnetic permeability $\mu$ is constant (see Ref. 18). In classical Faraday's law $\boldsymbol{f} = 0$ but, in theoretical physics, magnetic monopoles have been postulated by formal symmetry considerations (Bossavit[8]) and by reported observations (Cabrera[10]) so, for mathematical purposes, it may be interesting to consider $\boldsymbol{f} \neq 0$.

Here we shall consider nonlinear extensions of the classical Ohm's law in the form

$$\boldsymbol{e} = \rho\, \boldsymbol{j} \tag{1.2}$$

where the scalar resistivity $\rho = \rho(\theta, \boldsymbol{h}, \nabla\times\boldsymbol{h})$ can be taken as a highly nonlinear function of the temperature $\theta$ and of the magnetic field $\boldsymbol{h}$.

In this work we are concerned with the natural boundary value problem associated with (1.1),

$$\boldsymbol{h}\cdot\boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{e}\times\boldsymbol{n} = \boldsymbol{g} \quad \text{on } \Gamma = \partial\Omega, \tag{1.3}$$

where $\boldsymbol{n}$ denotes the outward unitary normal vector to $\Gamma$ and $\boldsymbol{g}$ is a tangential current field.

In order to take into account the thermal effects, in the nonisothermal stationary case, we have the equilibrium of energy

$$\nabla\cdot\boldsymbol{q} = \boldsymbol{j}\cdot\boldsymbol{e}, \tag{1.4}$$

where the heat flux $\boldsymbol{q} = -k\,\nabla\theta$ is given in terms of a nonlinear thermal conductivity $k = k(\theta)|\nabla\theta|^{q-2}$, $q > 1$, (the case of a constant $k > 0$ corresponds to the usual linear Fourier law) and the right-hand side of (1.4) represents the Joule heating.

We are particularly interested in the case of a nonlinear resistivity in (1.2) also of power type, $\rho = \nu(\theta)|\nabla\times\boldsymbol{h}|^{p-2}$, $p > 1$, when the equilibrium equations for $\boldsymbol{h}$ and $\theta$ may be written, from (1.1), (1.2) and (1.4), in the form

$$\nabla\times\big(\nu(\theta)|\nabla\times\boldsymbol{h}|^{p-2}\nabla\times\boldsymbol{h}\big) = \boldsymbol{f} \quad \text{and} \quad \nabla\cdot\boldsymbol{h} = 0 \quad \text{in } \Omega, \tag{1.5}$$

$$-\nabla\cdot\big(k(\theta)|\nabla\theta|^{q-2}\nabla\theta\big) = \nu(\theta)|\nabla\times\boldsymbol{h}|^{p} \quad \text{in } \Omega. \tag{1.6}$$

From the first equation of (1.5) the external field $\boldsymbol{f}$ must satisfy $\nabla\cdot\boldsymbol{f} = 0$ and, if we associate the second boundary condition of (1.3), the given field $\boldsymbol{g}$ should be tangential on $\Gamma$ and compatible with $\boldsymbol{f}$:

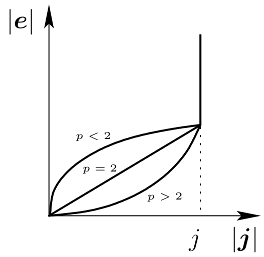
$$\nabla_\Gamma\cdot\boldsymbol{g} = \boldsymbol{f}\cdot\boldsymbol{n} \quad \text{on } \Gamma, \tag{1.7}$$

where $\nabla_\Gamma\cdot$ denotes the surface divergence. The presence of the external vector field $\boldsymbol{f}$, although unusual in classical Maxwell's theory, has been considered in the mathematical literature (see, for instance, Yin[30]) and, combined with the tangential current field on the boundary, it enlights the compatibility condition (1.7) – see also Remark 2.2 below.

We observe that the system (1.5) has a mathematical structure similar to the usual $p$-Laplace equation. Indeed, it reduces to a scalar equation with the left-hand side structurally similar to the left-hand side of (1.6) in only two dimensions, when the domain $\Omega = \omega\times\mathbb{R}$ is a longitudinal media and the parallel field $\boldsymbol{h} = \big(0, 0, h(x_1, x_2)\big)$, for $(x_1, x_2) \in \omega$. In this case, the second equation in (1.3) reduces to a Neumann boundary condition on $\partial\omega$ for the third component $h$.

In the evolutionary situation, this scalar coupled 2D problem has been considered by several authors, for instance, with $p = q = 2$ and Dirichlet data, in Rodrigues[28] or in Parietti-Rappaz[26] for alternating currents. Also for unsteady cases with $p = q = 2$ the 3D "induction heating" problem has been considered, for instance, in Bossavit-Rodrigues[9] or in Yin,[31] and with phase changes in Bermudez et al[5] or Manoranjan et al.[22]

A second example of a generalization of the constitutive law (1.2) arises in type-II superconductors and is known as an extension of the Bean critical-state model, in which the current density cannot exceed some given critical value $j > 0$.




When this critical threshold $j$ may vary with the absolute value $|\boldsymbol{h}|$ of the magnetic field, Prigozhin[27] has remarked that this model admits a formulation in terms of a quasi-variational inequality. Here we shall consider the case

$$\boldsymbol{e} = \begin{cases} \nu_0 |\nabla\times\boldsymbol{h}|^{p-2}\nabla\times\boldsymbol{h} & \text{if } |\nabla\times\boldsymbol{h}| < j(|\boldsymbol{h}|), \\ \\ \big(\nu_0\, j^{p-2} + \lambda\big)\nabla\times\boldsymbol{h} & \text{if } |\nabla\times\boldsymbol{h}| = j(|\boldsymbol{h}|), \end{cases} \tag{1.8}$$

with the nondegeneracy parameter $\nu_0 > 0$, where $\nu_0$ is a small given constant and $\lambda = \lambda(x) \geq 0$ can be regarded as a (unknown) Lagrange multiplier associated with the inequality constraint

$$|\nabla\times\boldsymbol{h}|(x) \leq j\big(|\boldsymbol{h}|(x)\big) \text{ a.e. } x \in \Omega. \tag{1.9}$$

The support of $\lambda$ lies in the superconductivity region

$$S = \big\{x \in \Omega : |\nabla\times\boldsymbol{h}|(x) = j\big(|\boldsymbol{h}|(x)\big)\big\}\,.$$

Multiplying the second equation of (1.1) by $\boldsymbol{v} - \boldsymbol{h}$ and integrating over $\Omega$ we obtain

$$\int_\Omega \nabla\times\boldsymbol{e}\cdot(\boldsymbol{v}-\boldsymbol{h}) = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h}).$$

Integrating by parts and using (1.3) we get

$$\int_\Omega \boldsymbol{e}\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}) = \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h}) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h}),$$

and by (1.8), we have

$$\int_\Omega \boldsymbol{e}\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}) = \int_\Omega \nu_0|\nabla\times\boldsymbol{h}|^{p-2}\nabla\times\boldsymbol{h}\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}) + \int_S \lambda\nabla\times\boldsymbol{h}\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}).$$

Choosing $\boldsymbol{v}$ such that $|\nabla\times\boldsymbol{v}| \leq j(|\boldsymbol{h}|)$, we have

$$\lambda\,\nabla\times\boldsymbol{h}\cdot\nabla\times\boldsymbol{v} \leq \lambda\,|\nabla\times\boldsymbol{h}|\,j(|\boldsymbol{h}|) = \lambda|\nabla\times\boldsymbol{h}|^2 \text{ in } S$$

obtaining the quasi-variational inequality

$$\int_\Omega \nu_0|\nabla\times\boldsymbol{h}|^{p-2}\nabla\times\boldsymbol{h}\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}) \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h}) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h}), \tag{1.10}$$

for any test function $\boldsymbol{v}$ such that $|\nabla\times\boldsymbol{v}| \leq j(|\boldsymbol{h}|)$.

This variational formulation for the analysis of critical state models for type-II superconductivity has stimulated the study of the power law model, since it has been shown to be the limit case $p \to \infty$, first in the scalar case, Barrett-Prigozhin,[3] and also in the case of a bounded simply-connected domain in $\mathbb{R}^3$ with null tangential component on the boundary, Yin et al,[32] both for the evolutionary problems.

In previous works of Yin and co-workers, instead of the natural boundary condition (1.3) for perfectly conductive (superconductive) walls ($\boldsymbol{h}$ tangential to the boundary: $\boldsymbol{h}\cdot\boldsymbol{n} = 0$ on $\Gamma$) only the case of perfectly permeable walls ($\boldsymbol{h}$ normal to the boundary: $\boldsymbol{h}\times\boldsymbol{n} = 0$ on $\Gamma$) has been considered. In particular, Yin and co-workers[32,33] obtained existence and some regularity results for this later evolutionary case for $p > 2$.

A main difficulty arises in extending the "main inequality of 3D vector analysis" (see Ref. 1 for $p = 2$) to the nonlinear framework $p \neq 2$. So the variational framework for magnetostatic and electrostatic problems that has been developed for the linear problems in the hilbertian framework (see, for instance, Refs. 14, 11, 15 and 12) requires, to the nonlinear case, further extensions that go beyond the natural extension to the $L^p$-integrable functions that has been done for arbitrary Lipschitz domains in $\mathbb{R}^3$ (see Refs. 23 and 24).

In this first work, that is restricted to steady-state problems, we develop the variational approach to power law Maxwell models of the type (1.5) with homogeneous normal trace (1.3) in bounded simply connected domains of $\mathbb{R}^3$. In Section 2 we show a new Poincar-Friedrichs inequality for $p > \frac{6}{5}$, extending a well-known property for $p = 2$, for vector functions with curl in $\boldsymbol{L}^p(\Omega)$ and with null divergence and null normal trace.

In Section 3 we develop the well-posed variational theory in two applicable directions: a strongly continuous dependence result and the limit variational inequality problem with bounded curl when the power of the nonlinearity $n \to \infty$. The stationary electromagnetic induction heating problem (1.5)-(1.6) is formulated and solved, in Section 4, as a coupled system for the cases $p > \frac{6}{5}$ and $q > \frac{5}{3}$.

Finally, in Section 5, we solve the quasi-variational inequality (1.9)-(1.10), by extending a continuous dependence result of Mosco type (see also Ref. 2) for convex sets of bounded curl and generalizing an existence result of Kunze-Rodrigues[17] in the scalar case.

## 2. The variational approach

### 2.1. *Weak formulation of the model problem*

In this section we consider an abstract problem associated with (1.5), for a given temperature, with the boundary conditions (1.3),

$$\nabla\times(\boldsymbol{a}(x,\nabla\times\boldsymbol{h}))=\boldsymbol{f} \quad \text{in} \quad \Omega, \tag{2.1a}$$

$$\nabla\cdot\boldsymbol{h}=0 \quad \text{in} \quad \Omega, \tag{2.1b}$$

$$\boldsymbol{a}(x,\nabla\times\boldsymbol{h})\times\boldsymbol{n}=\boldsymbol{g} \quad \text{on} \quad \Gamma, \tag{2.1c}$$

$$\boldsymbol{h}\cdot\boldsymbol{n}=0 \quad \text{on} \quad \Gamma, \tag{2.1d}$$

where $\boldsymbol{a}:\Omega\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ is a Carathodory function satisfying the structural conditions

$$\boldsymbol{a}(x,\boldsymbol{u})\cdot\boldsymbol{u}\geq a_*|\boldsymbol{u}|^p, \tag{2.2a}$$

$$|\boldsymbol{a}(x,\boldsymbol{u})|\leq a^*|\boldsymbol{u}|^{p-1}, \tag{2.2b}$$

$$\big(\boldsymbol{a}(x,\boldsymbol{u})-\boldsymbol{a}(x,\boldsymbol{v})\big)\cdot\big(\boldsymbol{u}-\boldsymbol{v}\big)>0, \text{ if } \boldsymbol{u}\neq\boldsymbol{v}, \tag{2.2c}$$

for given constants $a_*$, $a^*>0$, for all $\boldsymbol{u}$, $\boldsymbol{v}\in\mathbb{R}^3$ and a.e. $x\in\Omega$.

The variational approach of this problem leads to introduce

$$\mathbb{W}^p(\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega):\nabla\cdot\boldsymbol{v}=0\,,\boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma}=0\big\}\,,\quad 1<p<\infty, \tag{2.3}$$

which is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$. Here $\boldsymbol{W}^{1,p}(\Omega)=W^{1,p}(\Omega)^3$ denotes the usual Sobolev space, where the following Green-type formula

$$\int_\Omega\nabla\times\boldsymbol{v}\cdot\boldsymbol{\varphi}-\int_\Omega\boldsymbol{v}\cdot\nabla\times\boldsymbol{\varphi}=\int_\Gamma\boldsymbol{v}\times\boldsymbol{n}\cdot\boldsymbol{\varphi}\quad\forall\boldsymbol{\varphi}\in\boldsymbol{W}^{1,p'}(\Omega) \tag{2.4}$$

holds with $\boldsymbol{v}\times\boldsymbol{n}_{|\Gamma}\in\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ in the sense of traces (see Refs. 13 and 23).

From (2.1) we are then naturally lead to the weak formulation of the model problem: given $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{g}\in\boldsymbol{L}^{p'}(\Gamma)$, find $\boldsymbol{h}\in\mathbb{W}^p(\Omega)$ such that

$$\int_\Omega\boldsymbol{a}(x,\nabla\times\boldsymbol{h})\cdot\nabla\times\boldsymbol{\varphi}=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{\varphi}+\int_\Gamma\boldsymbol{g}\cdot\boldsymbol{\varphi}\quad\forall\boldsymbol{\varphi}\in\mathbb{W}^p(\Omega). \tag{2.5}$$

### 2.2. *The space* $\mathbb{W}^p(\Omega)$

In this section we characterize the space $\mathbb{W}^p(\Omega)$ and we prove an essential extension, for $p>\frac{6}{5}$, of the Poincar-Friedrichs inequality for the well-known case $p=2$. The following assumption stands in the rest of this work:

$$\Omega \text{ is a bounded domain in } \mathbb{R}^3, \text{ simply connected, with a } \mathscr{C}^2 \text{ boundary } \Gamma. \tag{2.6}$$

Here we also define the $\boldsymbol{W}^{1,p}$-norm of $\boldsymbol{v}$, as usually, by the sum of the $\boldsymbol{L}^p$-norms of $\boldsymbol{v}$ and $\nabla\boldsymbol{v}$.

**Theorem 2.1.** *In $\mathbb{W}^p(\Omega)$, for $\Omega$ satisfying (2.6) and $p>\frac{6}{5}$, the semi-norm $\|\nabla\times\cdot\|_{\boldsymbol{L}^p(\Omega)}$ is a norm, denoted by $\|\cdot\|_{\mathbb{W}^p(\Omega)}$, equivalent to the $\boldsymbol{W}^{1,p}$-norm.*

First we introduce the functional spaces

$$\boldsymbol{W}^p(\nabla\cdot,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega):\ \nabla\cdot\boldsymbol{v}\in L^p(\Omega)\big\}$$

and

$$\boldsymbol{W}^p(\nabla\times,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega):\ \nabla\times\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)\big\},$$

endowed with the graph norms, in particular

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^p(\nabla\times,\Omega)}=\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}+\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\,.$$

$\boldsymbol{W}^p_0(\nabla\cdot,\Omega)$ and $\boldsymbol{W}^p_0(\nabla\times,\Omega)$ represent the closure of $\mathscr{D}(\Omega)^3$ in the spaces $\boldsymbol{W}^p(\nabla\cdot,\Omega)$ and $\boldsymbol{W}^p(\nabla\times,\Omega)$, respectively. All these spaces are separable reflexive Banach spaces when $1<p<\infty$ and $\mathscr{D}(\bar\Omega)^3$ is dense in $\boldsymbol{W}^p(\nabla\cdot,\Omega)$ and $\boldsymbol{W}^p(\nabla\times,\Omega)$, where Green formulas, such as (2.4), hold, being the boundary integrals interpreted in the duality between $W^{\frac1p,p'}(\Gamma)$ and $W^{-\frac1p,p}(\Gamma)$ (see, for instance, Refs. 13 and 23).

The following characterizations

$$\boldsymbol{W}^p_0(\nabla\cdot,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\cdot,\Omega):\boldsymbol{v}\cdot\boldsymbol{n}_{|_\Gamma}=0\big\}$$

and

$$\boldsymbol{W}^p_0(\nabla\times,\Omega)=\big\{\boldsymbol{v}\in\boldsymbol{W}^p(\nabla\times,\Omega):\boldsymbol{v}\times\boldsymbol{n}_{|_\Gamma}=0\big\}$$

are also well-known.

**Lemma 2.1.** *With the same assumptions of the Theorem 2.1, in $\mathbb{W}^p(\Omega)$ the norm*

$$|\!|\!|\cdot|\!|\!|=\|\cdot\|_{\boldsymbol{L}^{p\vee2}(\Omega)}+\|\nabla\times\cdot\|_{\boldsymbol{L}^p(\Omega)}$$

*is equivalent to the $\boldsymbol{W}^{1,p}$-norm. Here $p\vee2=\max\{p,2\}$.*

**Proof.** We prove first that

$$\mathbb{W}^p(\Omega)=\boldsymbol{L}^2(\Omega)\cap\boldsymbol{W}^p(\nabla\times,\Omega)\cap\boldsymbol{W}^p_0(\nabla\cdot,0,\Omega).$$

where $\boldsymbol{W}^p_0(\nabla\cdot,0,\Omega)=\big\{v\in\boldsymbol{W}^p_0(\nabla\cdot,\Omega):\nabla\cdot v=0\big\}$.

Observing that for $p\geq\frac65$ we have $W^{1,p}(\Omega)\hookrightarrow L^2(\Omega)$, we need to prove that a divergence free function $\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\cap\boldsymbol{L}^p(\Omega)=\boldsymbol{L}^{2\vee p}(\Omega)$, having curl in $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{v}\cdot\boldsymbol{n}_{|_\Gamma}=0$, has gradient in $\boldsymbol{L}^p(\Omega)$.

Considering

$$\boldsymbol{V}=\big\{\boldsymbol{\xi}\in\boldsymbol{H}^1_0(\Omega):\nabla\cdot\boldsymbol{\xi}=0\big\},$$

the homogeneous Dirichlet problem for

$$-\Delta\boldsymbol{\xi}=\nabla\times(\nabla\times\boldsymbol{\xi})=\nabla\times\boldsymbol{v}\quad\text{in }\Omega \tag{2.7}$$

has a unique solution $\boldsymbol{\xi}\in\boldsymbol{V}$. Indeed, in the Hilbert space $\boldsymbol{V}$, the norms $\|\cdot\|_{\boldsymbol{H}^1(\Omega)}$ and $\|\nabla\times\cdot\|_{\boldsymbol{L}^2(\Omega)}$ are equivalent (see Ref. 13, p. 209) and, since $\nabla\times\boldsymbol{v}\in\boldsymbol{L}^p(\Omega)$ and

$L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$ if $p \geq \frac{6}{5}$, the existence of solution to the weak formulation of (2.7)

$$\int_\Omega \nabla\times\boldsymbol{\xi}\cdot\nabla\times\boldsymbol{\varphi} = \int_\Omega \nabla\times\boldsymbol{v}\cdot\boldsymbol{\varphi}, \quad \forall\boldsymbol{\varphi}\in\boldsymbol{V},$$

follows from standard results. By elliptic regularity we may conclude that $\boldsymbol{\xi} \in \boldsymbol{H}^1_0(\Omega)\cap\boldsymbol{W}^{2,p}(\Omega)$ (see Ref. 16).

Let us consider the function $\boldsymbol{\psi} = \nabla\times\boldsymbol{\xi} - \boldsymbol{v}$. By the properties of $\boldsymbol{v}$ and (2.7) this function is such that

$$\begin{aligned}
&\boldsymbol{\psi}\in\boldsymbol{L}^p(\Omega)\cap\boldsymbol{L}^2(\Omega),\\
&\nabla\cdot\boldsymbol{\psi} = \nabla\cdot(\nabla\times\boldsymbol{\xi}) - \nabla\cdot\boldsymbol{v} = 0 \qquad \text{in} \quad \Omega,\\
&\nabla\times\boldsymbol{\psi} = \nabla\times(\nabla\times\boldsymbol{\xi}) - \nabla\times\boldsymbol{v} = 0 \quad \text{in} \quad \Omega,\\
&\boldsymbol{\psi}\cdot\boldsymbol{n} = (\nabla\times\boldsymbol{\xi})\cdot\boldsymbol{n} - \boldsymbol{v}\cdot\boldsymbol{n} = 0 \qquad \text{on} \quad \Gamma,
\end{aligned}$$

being this last equation a consequence of $\boldsymbol{\xi}=0$ on $\Gamma$, which implies that $\nabla\times\boldsymbol{\xi}_{|\Gamma}$ is orthogonal to $\boldsymbol{n}$, by applying Stokes theorem over $\Gamma$.

So, $\boldsymbol{\psi}\in\mathbb{H}(\Omega) = \left\{\boldsymbol{u}\in\boldsymbol{L}^2(\Omega) : \nabla\times\boldsymbol{u} = 0,\, \nabla\cdot\boldsymbol{u} = 0,\, \boldsymbol{u}\cdot\boldsymbol{n}_{|\Gamma} = 0\right\}$. As we have that $\dim\mathbb{H}(\Omega) = 0$ (see Ref. 13, p. 219), we conclude that $\boldsymbol{\psi} = 0$, and so $\boldsymbol{v} = \nabla\times\boldsymbol{\xi}\in\boldsymbol{W}^{1,p}(\Omega)$.

To prove the equivalence of the norms $\|\cdot\|_{\boldsymbol{W}^{1,p}(\Omega)}$ and $\|\cdot\|_{\boldsymbol{L}^{p\vee 2}(\Omega)} + \|\nabla\times\cdot\|_{\boldsymbol{L}^p(\Omega)}$ in $\mathbb{W}^p(\Omega)$, it is enough to observe that this space is a Banach space with both norms and that the first norm is stronger than the second one. □

For the proof of Theorem 2.1 we also need the following Lemma, which proof can be found in Lions-Magenes,[21] p. 171.

**Lemma 2.2 (Peetre).** *Let $E_0$, $E_1$ and $E_2$ be three Banach spaces and let*

$$A_1 : E_0 \longrightarrow E_1 \text{ and } A_2 : E_0 \longrightarrow E_2$$

*be two linear continuous mappings with:*

1. *$A_2$ is a compact mapping;*
2. *there exists a constant $c > 0$ such that*

$$\|v\|_{E_0} \leq c\left(\|A_1 v\|_{E_1} + \|A_2 v\|_{E_2}\right), \ \forall v\in E_0. \tag{2.8}$$

*Then:*

1. $\ker A_1$ *has finite dimension and* $\operatorname{Im} A_1$ *is closed;*
2. *there exists a constant $C_0 > 0$ such that*

$$\inf_{w\in\ker A_1}\|v+w\|_{E_0} \leq C_0\|A_1 v\|_{E_1}.$$

**Proof of Theorem 2.1:** To see that the semi-norm $\|\nabla\times\cdot\|_{\boldsymbol{L}^p(\Omega)}$ is a norm in $\mathbb{W}^p(\Omega)$ for $p > \frac{6}{5}$, we define the linear continuous operators

$$\begin{aligned}
A_1 : \mathbb{W}^p(\Omega) &\longrightarrow \boldsymbol{L}^p(\Omega)\\
\boldsymbol{v} &\longmapsto \nabla\times\boldsymbol{v}
\end{aligned}$$

and

$$\begin{array}{rcl} A_2 : \mathbb{W}^p(\Omega) & \longrightarrow & \boldsymbol{L}^2(\Omega) \cap \boldsymbol{L}^p(\Omega), \\ \boldsymbol{v} & \longmapsto & \boldsymbol{v} \end{array}$$

noting that $A_2$ is compact.

Since

$$|\!|\!| \boldsymbol{v} |\!|\!| = \|A_1 \boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|A_2 \boldsymbol{v}\|_{\boldsymbol{L}^{p\vee 2}(\Omega)},$$

the inequality (2.8) is verified, so by Peetre lemma,

$$\exists\, C_0 > 0 \qquad \inf_{\boldsymbol{u} \in \ker A_1} |\!|\!| \boldsymbol{u} + \boldsymbol{v} |\!|\!| \leq C_0 \|A_1 \boldsymbol{v}\|_p . \tag{2.9}$$

Observing that $\ker A_1 \subseteq \mathbb{H}(\Omega) = 0$, the relation (2.9) is equivalent to

$$\exists\, C_0 > 0 \quad |\!|\!| \boldsymbol{v} |\!|\!| \leq C_0 \|\nabla \times \boldsymbol{v}\|_p ,$$

proving that the semi-norm $\|\nabla \times \cdot\|_p$ is a norm in $\mathbb{W}^p(\Omega)$, equivalent to the $\boldsymbol{W}^{1,p}$-norm. □

**Corollary 2.1.** *Let $\Omega$ satisfy (2.6) and $p > \frac{6}{5}$. Given $\boldsymbol{h} \in \mathbb{W}^p(\Omega)$, its trace, $\boldsymbol{h}_{|\Gamma} \in \boldsymbol{W}^{\frac{1}{p'},p}(\Gamma)$, is such that*

$$\left\|\boldsymbol{h}_{|\Gamma}\right\|_{\boldsymbol{W}^{\frac{1}{p'},p}(\Gamma)} \leq C \|\boldsymbol{h}\|_{\mathbb{W}^p(\Omega)} \tag{2.10}$$

*for some $C = C(\Omega, p) > 0$.*

**Proof.** This result is an immediate consequence of the continuity of the trace map from $\boldsymbol{W}^{1,p}(\Omega)$ onto $\boldsymbol{W}^{\frac{1}{p'},p}(\Gamma)$ and of the equivalence, in $\mathbb{W}^p(\Omega)$, between the $\boldsymbol{W}^{1,p}$ and $\mathbb{W}^p$ norms. □

### 2.3. *Solution of the variational problem*

We can now show the existence and uniqueness of solution of the variational formulation (2.5) of the problem (2.1).

**Proposition 2.1.** *For $\Omega$ satisfying (2.6) and $p > \frac{6}{5}$, the problem (2.5) has a unique solution.*

**Proof.** $\mathbb{W}^p(\Omega)$ is a separable reflexive Banach space. Considering the operator $A : \mathbb{W}^p(\Omega) \longrightarrow \mathbb{W}^p(\Omega)'$ such that

$$\langle A\boldsymbol{h}, \boldsymbol{\varphi}\rangle = \int_\Omega \boldsymbol{a}(x, \nabla \times \boldsymbol{h}) \cdot \nabla \times \boldsymbol{\varphi} \quad \forall \boldsymbol{h},\, \boldsymbol{\varphi} \in \mathbb{W}^p(\Omega),$$

the structural properties (2.2) allow us to conclude that $A$ is a bounded, hemicontinuous, monotone and coercive operator.

Defining $L \in \mathbb{W}^p(\Omega)'$ such that

$$\langle L, \boldsymbol{\varphi}\rangle = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{\varphi} \quad \forall \boldsymbol{\varphi} \in \mathbb{W}^p(\Omega),$$

a well-known existence theorem to monotone operators (Ref. 20) guarantees the existence of solution in $\mathbb{W}^p(\Omega)$ to the problem $A\boldsymbol{h} = L$. The uniqueness results directly from the strict monotonicity (2.2c) of the operator $A$. □

**Remark 2.1.** For the existence of solution of the strong boundary value problem (2.1) with given data $(\boldsymbol{f}, \boldsymbol{g})$, as we observed in the introduction, it is necessary that $\boldsymbol{f}$ is divergence free and $\boldsymbol{g}$ is tangential on $\Gamma$. However the weak formulation (2.5) of the problem (2.1) has a unique solution with no restrictions on the data, that can be taken more generally as an element $L \in \mathbb{W}^p(\Omega)'$ in the right-hand side of (2.5). Usually, a solution of a weak problem is also solution of the strong one, as long as it has enough regularity. The situation here is different. In fact, the compatibility condition (1.7) is a necessary condition for the existence of solution of (2.1) (see Ref. 23, p. 143) but not for the existence of a weak solution.

**Remark 2.2.** Let us observe that, given $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, using the Helmholtz decomposition (see Ref. 29), we may write $\boldsymbol{f} = \boldsymbol{f}_0 + \nabla\xi$, where $\boldsymbol{f}_0$ is divergence free and, given $\boldsymbol{g} \in \boldsymbol{L}^{p'}(\Gamma)$, we may write $\boldsymbol{g} = \boldsymbol{g}_T + \boldsymbol{g}_N$, where $\boldsymbol{g}_T$ and $\boldsymbol{g}_N$ are, respectively, the tangential and the normal components of $\boldsymbol{g}$.

Since the set of test functions $\mathbb{W}^p(\Omega)$ only takes into account $\boldsymbol{f}_0$ (the divergence free component of $\boldsymbol{f}$) and $\boldsymbol{g}_T$ (the tangential component of $\boldsymbol{g}$), the problems (2.5) with data $(\boldsymbol{f}, \boldsymbol{g})$ and $(\boldsymbol{f}_0, \boldsymbol{g}_T)$ are the same and both correspond to the weak formulation of problem (2.1) with data $(\boldsymbol{f}_0, \boldsymbol{g}_T)$.

Although, because of the structure of the test functions in the weak formulation of the problem, in order to go back to the strong formulation, we have to impose the compatibility condition (1.7). In fact, if $\boldsymbol{h}$ is a regular solution of (2.5) with data $(\boldsymbol{f}, \boldsymbol{g})$, if we set $\boldsymbol{a} = \boldsymbol{a}(x, \nabla\times\boldsymbol{h})$, integrating by parts, we have

$$\int_\Omega \big(\nabla\times\boldsymbol{a} - \boldsymbol{f}_0\big)\cdot\boldsymbol{\varphi} + \int_\Gamma \big(\boldsymbol{a}\times\boldsymbol{n} - \boldsymbol{g}_T\big)\cdot\boldsymbol{\varphi} = 0 \quad \forall\boldsymbol{\varphi} \in \mathbb{W}^p(\Omega),$$

so, $\boldsymbol{a}\times\boldsymbol{n} = \boldsymbol{g}_T$ on $\Gamma$. Noticing that the function $\boldsymbol{\zeta} = \nabla\times\boldsymbol{a} - \boldsymbol{f}_0$ is divergence free and, on the other hand, since $\boldsymbol{a}\times\boldsymbol{n}_{|\Gamma}$ is tangential, we have, on $\Gamma$

$$\boldsymbol{\zeta}\cdot\boldsymbol{n} = \big(\nabla\times\boldsymbol{a}\cdot\boldsymbol{n}\big) - \boldsymbol{f}_0\cdot\boldsymbol{n} = \nabla_\Gamma\cdot\big(\boldsymbol{a}\times\boldsymbol{n}\big) - \boldsymbol{f}_0\cdot\boldsymbol{n} = \nabla_\Gamma\cdot\boldsymbol{g} - \boldsymbol{f}_0\cdot\boldsymbol{n} = 0,$$

as long as the compatibility condition (1.7) is satisfied. This implies that $\boldsymbol{\zeta}$ is zero and (2.1a) is satisfied.

## 3. Properties of the variational solution

### 3.1. *A continuous dependence result*

Consider sequences of functions $\boldsymbol{f}_n \in \boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{g}_n \in \boldsymbol{L}^{p'}(\Gamma)$ and a sequence of Carathodory functions $\boldsymbol{a}_n$ satisfying, together with $\boldsymbol{a}$, the properties (2.2) with the same $a_*$ and $a^*$, where we replace (2.2c) by

$$\big(\boldsymbol{a}(x,\boldsymbol{u}) - \boldsymbol{a}(x,\boldsymbol{v})\big)\cdot(\boldsymbol{u}-\boldsymbol{v}) \geq \begin{cases} a_*|\boldsymbol{u}-\boldsymbol{v}|^p, & p \geq 2, \\ a_*(|\boldsymbol{u}| + |\boldsymbol{v}|)^{p-2}|\boldsymbol{u}-\boldsymbol{v}|^2, & 1 < p < 2. \end{cases} \tag{3.1}$$

Let, for each $n \in \mathbb{N}$, $\boldsymbol{h}_n \in \mathbb{W}^p(\Omega)$ be the solution of the problem

$$\int_\Omega a_n(x, \nabla\times\boldsymbol{h}_n)\cdot\nabla\times\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f}_n\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}_n\cdot\boldsymbol{\varphi} \quad \forall\boldsymbol{\varphi}\in\mathbb{W}^p(\Omega). \tag{3.2}$$

**Theorem 3.1.** *If* $\boldsymbol{f}_n \xrightarrow[n]{} \boldsymbol{f}$ *in* $\boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{g}_n \xrightarrow[n]{} \boldsymbol{g}$ *in* $\boldsymbol{L}^{p'}(\Gamma)$ *and* $a_n \xrightarrow[n]{} a$ *a.e. in* $\Omega\times\mathbb{R}^3$, *then the sequence of the solutions of the problems* (3.2), $\{\boldsymbol{h}_n\}_n$, *converges in* $\mathbb{W}^p(\Omega)$ *to* $\boldsymbol{h}$, *solution of the problem* (2.5).

**Proof.** Replacing in (2.5) and (3.2) the test function $\boldsymbol{\varphi}$ by $\boldsymbol{h}_n - \boldsymbol{h}$ we have that

$$\int_\Omega \big(\boldsymbol{a}_n(x,\nabla\times\boldsymbol{h}_n) - \boldsymbol{a}(x,\nabla\times\boldsymbol{h})\big)\cdot\nabla\times(\boldsymbol{h}_n-\boldsymbol{h}) = \int_\Omega (\boldsymbol{f}_n-\boldsymbol{f})\cdot(\boldsymbol{h}_n-\boldsymbol{h}) + \int_\Gamma (\boldsymbol{g}_n-\boldsymbol{g})\cdot(\boldsymbol{h}_n-\boldsymbol{h})$$

and so $\boldsymbol{h}$ and $\boldsymbol{h}_n$ satisfy the relation

$$\int_\Omega \big(\boldsymbol{a}_n(x,\nabla\times\boldsymbol{h}_n) - \boldsymbol{a}_n(x,\nabla\times\boldsymbol{h})\big)\cdot\nabla\times(\boldsymbol{h}_n-\boldsymbol{h}) + \int_\Omega \big(\boldsymbol{a}_n(x,\nabla\times\boldsymbol{h}) - \boldsymbol{a}(x,\nabla\times\boldsymbol{h})\big)\cdot\nabla\times(\boldsymbol{h}_n-\boldsymbol{h}) = \int_\Omega (\boldsymbol{f}_n-\boldsymbol{f})\cdot(\boldsymbol{h}_n-\boldsymbol{h}) + \int_\Gamma (\boldsymbol{g}_n-\boldsymbol{g})\cdot(\boldsymbol{h}_n-\boldsymbol{h}). \tag{3.3}$$

Notice that, given $\boldsymbol{f}\in\boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{g}\in\boldsymbol{L}^{p'}(\Gamma)$ and $\boldsymbol{h}\in\mathbb{W}^p(\Omega)$ there are constants $C_1$ and $C_2$ such that

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{h} \le \|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{h}\|_{\boldsymbol{L}^p(\Omega)} \le C_1\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{h}\|_{\mathbb{W}^p(\Omega)}$$

and

$$\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{h} \le \|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)}\|\boldsymbol{h}\|_{\boldsymbol{L}^p(\Gamma)} \le C_2\|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)}\|\boldsymbol{h}\|_{\mathbb{W}^p(\Omega)}.$$

Using the property (3.1), in the case $p\ge 2$, from the relation (3.3) we obtain

$$a_*\|\nabla\times(\boldsymbol{h}_n-\boldsymbol{h})\|^p_{\boldsymbol{L}^p(\Omega)} \le \|\boldsymbol{a}_n(x,\nabla\times\boldsymbol{h}) - \boldsymbol{a}(x,\nabla\times\boldsymbol{h})\|_{\boldsymbol{L}^{p'}(\Omega)}\|\nabla\times(\boldsymbol{h}_n-\boldsymbol{h})\|_{\boldsymbol{L}^p(\Omega)} + C_1\|\boldsymbol{f}_n-\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}\|\boldsymbol{h}_n-\boldsymbol{h}\|_{\mathbb{W}^p(\Omega)} + C_2\|\boldsymbol{g}_n-\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)}\|\boldsymbol{h}_n-\boldsymbol{h}\|_{\mathbb{W}^p(\Omega)}$$

and so

$$a_*\|\nabla\times(\boldsymbol{h}_n-\boldsymbol{h})\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \le \|\boldsymbol{a}_n(x,\nabla\times\boldsymbol{h}) - \boldsymbol{a}(x,\nabla\times\boldsymbol{h})\|_{\boldsymbol{L}^{p'}(\Omega)} + C_1\|\boldsymbol{f}_n-\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + C_2\|\boldsymbol{g}_n-\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)}. \tag{3.4}$$

In the case $\frac{6}{5} < p < 2$ using the property (3.1), from (3.3) we obtain

$$a_* \int_\Omega \big(|\nabla\times \boldsymbol{h}_n| + |\nabla\times \boldsymbol{h}|\big)^{p-2} \big|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\big|^2 \leq \\ \|\boldsymbol{a}_n(x,\nabla\times \boldsymbol{h}) - \boldsymbol{a}(x,\nabla\times \boldsymbol{h})\|_{\boldsymbol{L}^{p'}(\Omega)} \|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\|_{\boldsymbol{L}^p(\Omega)} + \\ C_1 \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} \|\boldsymbol{h}_n - \boldsymbol{h}\|_{\mathbb{W}^p(\Omega)} + C_2 \|\boldsymbol{g}_n - \boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)} \|\boldsymbol{h}_n - \boldsymbol{h}\|_{\mathbb{W}^p(\Omega)}. \quad (3.5)$$

Using a reverse Hlder inequality with $0 < \frac{p}{2} < 1$ and $\frac{p}{p-2} < 0$, we obtain

$$a_* \int_\Omega \big(|\nabla\times \boldsymbol{h}_n| + |\nabla\times \boldsymbol{h}|\big)^{p-2} \big|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\big|^2 \geq \\ a_* \Big(2^{p-1}\Big( \|\nabla\times \boldsymbol{h}_n\|^p_{\boldsymbol{L}^p(\Omega)} + \|\nabla\times \boldsymbol{h}\|^p_{\boldsymbol{L}^p(\Omega)} \Big)\Big)^{\frac{p-2}{p}} \|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\|^2_{\boldsymbol{L}^p(\Omega)}. \quad (3.6)$$

We have, from (2.2a), that for $n \in \mathbb{N}$

$$a_* \|\nabla\times \boldsymbol{h}_n\|^p_{\boldsymbol{L}^p(\Omega)} = \int_\Omega a_* |\nabla\times \boldsymbol{h}_n|^p \leq \int_\Omega \boldsymbol{a}(x,\nabla\times \boldsymbol{h}_n)\cdot \nabla\times \boldsymbol{h}_n = \\ \int_\Omega \boldsymbol{f}_n \cdot \boldsymbol{h}_n + \int_\Gamma \boldsymbol{g}_n \cdot \boldsymbol{h}_n \leq \Big(C_1 \|\boldsymbol{f}_n\|_{\boldsymbol{L}^{p'}(\Omega)} + C_2 \|\boldsymbol{g}_n\|_{\boldsymbol{L}^{p'}(\Gamma)}\Big) \|\nabla\times \boldsymbol{h}_n\|_{\boldsymbol{L}^p(\Omega)}.$$

Using the convergences of $\boldsymbol{f}_n$ and $\boldsymbol{g}_n$ to $\boldsymbol{f}$ and $\boldsymbol{g}$ respectively, there exists a positive constant $C$ such that

$$a_* \|\nabla\times \boldsymbol{h}_n\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \leq C.$$

Going back to the inequality (3.6), this last estimate allow us to write

$$a_* \int_\Omega \big(|\nabla\times \boldsymbol{h}_n| + |\nabla\times \boldsymbol{h}|\big)^{p-2} \big|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\big|^2 \geq C_3 \|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\|^2_{\boldsymbol{L}^p(\Omega)}$$

for some positive constant $C_3$ independent of $n$.

Using this last inequality, from (3.5) we have

$$C_3 \|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\|_{\boldsymbol{L}^p(\Omega)} \leq \\ \|\boldsymbol{a}_n(x,\nabla\times \boldsymbol{h}) - \boldsymbol{a}(x,\nabla\times \boldsymbol{h})\|_{\boldsymbol{L}^{p'}(\Omega)} + \\ C_1 \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + C_2 \|\boldsymbol{g}_n - \boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)}. \quad (3.7)$$

We can write both relations (3.4) and (3.7) in the following inequality

$$\|\nabla\times(\boldsymbol{h}_n - \boldsymbol{h})\|^{(p-1)\vee 1}_{\boldsymbol{L}^p(\Omega)} \leq \\ C_4\Big( \|\boldsymbol{a}_n(x,\nabla\times \boldsymbol{h}) - \boldsymbol{a}(x,\nabla\times \boldsymbol{h})\|_{\boldsymbol{L}^{p'}(\Omega)} + \\ \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\boldsymbol{g}_n - \boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)}\Big) \quad (3.8)$$

valid for $p > \frac{6}{5}$, where $C_4$ is a positive constant independent of $n$. □

**Remark 3.1.** If

$$\boldsymbol{a}_n(x, \boldsymbol{u}) = \nu_n(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u} \quad \text{and} \quad \boldsymbol{a}(x, \boldsymbol{u}) = \nu(x)|\boldsymbol{u}|^{p-2}\boldsymbol{u}$$

from (3.8) we have a stronger result

$$\|\boldsymbol{h}_n - \boldsymbol{h}\|_{\mathbb{W}^p(\Omega)}^{(p-1)\vee 1} \leq C_p\Big( \|\nu_n - \nu\|_{L^\infty(\Omega)} + \|\boldsymbol{f}_n - \boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)} + \|\boldsymbol{g}_n - \boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)} \Big).$$

### 3.2. *A limit problem when* $n \to \infty$

In this section we assume that $\frac{6}{5} < p < n$, $p'$ and $n'$ are respectively the conjugate exponents of $p$ and $n$, $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{p'}(\Gamma)$. For simplicity we denote

$$\nabla_p \times \boldsymbol{u} = |\nabla \times \boldsymbol{u}|^{p-2} \nabla \times \boldsymbol{u}. \tag{3.9}$$

Since the operator $A_n : \mathbb{W}^n(\Omega) \longrightarrow \mathbb{W}^n(\Omega)'$ defined by

$$\langle A_n \boldsymbol{u}, \boldsymbol{v}\rangle = \int_\Omega \nabla_n \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} + \int_\Omega \boldsymbol{\delta}(x, \nabla \times \boldsymbol{u}) \cdot \nabla \times \boldsymbol{v},$$

where $\boldsymbol{\delta}(x, \boldsymbol{u})$ satisfying (2.2 a,b) and (3.1) with nonnegative constants $\delta_*$ and $\delta^*$ (in particular, we may have $\delta_* = \delta^* = 0$), is bounded, hemicontinuous, strictly monotone and coercive, the problem

$$\int_\Omega \nabla_n \times \boldsymbol{h}_n \cdot \nabla \times \boldsymbol{\varphi} + \int_\Omega \boldsymbol{\delta}(x, \nabla \times \boldsymbol{h}_n) \cdot \nabla \times \boldsymbol{\varphi} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{\varphi} \quad \forall \boldsymbol{\varphi} \in \mathbb{W}^n(\Omega), \tag{3.10}$$

has a unique solution $\boldsymbol{h}_n$.

We are going to characterize the limit of $\{\boldsymbol{h}_n\}_n$, when $n \to \infty$. If we take $\boldsymbol{\varphi} = \boldsymbol{h}_n$ in (3.10) we obtain

$$\int_\Omega |\nabla \times \boldsymbol{h}_n|^n + \int_\Omega \boldsymbol{\delta}(x, \nabla \times \boldsymbol{h}_n) \cdot \nabla \times \boldsymbol{h}_n = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{h}_n + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{h}_n$$

and so there exists a positive constant $C_1$, independent of $n$, such that

$$\int_\Omega |\nabla \times \boldsymbol{h}_n|^n + \delta_* \int_\Omega |\nabla \times \boldsymbol{h}_n|^p \leq \\ C_1\Big( \|\boldsymbol{f}\|_{\boldsymbol{L}^{n'}(\Omega)} \|\boldsymbol{h}_n\|_{\boldsymbol{L}^n(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{n'}(\Gamma)} \|\boldsymbol{h}_n\|_{\boldsymbol{W}^{\frac{1}{n'},n}(\Gamma)} \Big). \tag{3.11}$$

Using the equivalence between the $\mathbb{W}^n$ and $\boldsymbol{W}^{1,n}$ norms and the inequality of Corollary 2.1, we obtain the *a priori* estimate

$$\|\nabla \times \boldsymbol{h}_n\|_{\boldsymbol{L}^n(\Omega)} \leq C_2, \qquad C_2 = C_2\big(\|\boldsymbol{f}\|_{\boldsymbol{L}^{p'}(\Omega)}, \|\boldsymbol{g}\|_{\boldsymbol{L}^{p'}(\Gamma)}\big). \tag{3.12}$$

Defining the convex set $\mathbb{K}_\infty = \big\{\boldsymbol{v} \in \mathbb{W}^\infty(\Omega) : |\nabla \times \boldsymbol{v}| \leq 1 \text{ a.e. in } \Omega\big\}$, where in (2.3), letting $p = \infty$, we obtain the definition of $\mathbb{W}^\infty(\Omega)$, we consider the following variational inequality.

To find $\boldsymbol{h}_\infty \in \mathbb{K}_\infty$ such that

$$\int_\Omega \boldsymbol{\delta}(x, \nabla\times\boldsymbol{h}_\infty)\cdot\nabla\times(\boldsymbol{\varphi}-\boldsymbol{h}_\infty) \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{\varphi}-\boldsymbol{h}_\infty)+\int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{\varphi}-\boldsymbol{h}_\infty) \qquad \forall\boldsymbol{\varphi} \in \mathbb{K}_\infty. \tag{3.13}$$

**Theorem 3.2.** *Assuming that the operator* $\boldsymbol{\delta}$ *satisfies* (2.2 a,b)*, with nonnegative* $\delta_*$ *and* $\delta^*$*, and* (3.1)*, let* $\boldsymbol{h}_n$ *denote the solution of* (3.10)*. Then we have, at least for subsequences,*

$$\boldsymbol{h}_n \rightharpoonup \boldsymbol{h}_\infty \qquad \text{in } \mathbb{W}^p(\Omega)\text{-weak}, \tag{3.14}$$

*where* $\boldsymbol{h}_\infty$ *is a solution of* (3.13)*.*

*If* $\delta_* > 0$*, the whole sequence converges to the unique limit.*

**Proof.** As the sequence $\{\boldsymbol{h}_n\}_n$ is bounded in $\mathbb{W}^p(\Omega)$, there exists a function $\boldsymbol{h}_\infty$ such that at least for a subsequence, still denoted by $\{\boldsymbol{h}_n\}_n$, we have

$$\boldsymbol{h}_n \rightharpoonup \boldsymbol{h}_\infty \qquad \text{weakly in } \mathbb{W}^p(\Omega).$$

Due to the equivalence between the norm of $\mathbb{W}^p(\Omega)$ and of $\boldsymbol{W}^{1,p}(\Omega)$, we may say that

$$\begin{aligned} \nabla\times\boldsymbol{h}_n &\rightharpoonup \nabla\times\boldsymbol{h}_\infty && \text{in } \boldsymbol{L}^p(\Omega)\text{-weak},\\ \boldsymbol{h}_n &\longrightarrow \boldsymbol{h}_\infty && \text{in } \boldsymbol{L}^p(\Omega)\text{-strong}. \end{aligned}$$

Given $p < q < n$ we have, using the estimate (3.12),

$$\|\nabla\times\boldsymbol{h}_n\|_{\boldsymbol{L}^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{n}}\,\|\nabla\times\boldsymbol{h}_n\|_{\boldsymbol{L}^n(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{n}} C_2^{\frac{1}{n}}$$

and so

$$\|\nabla\times\boldsymbol{h}_\infty\|_{\boldsymbol{L}^q(\Omega)} \leq \liminf_n \|\nabla\times\boldsymbol{h}_n\|_{\boldsymbol{L}^q(\Omega)} \leq |\Omega|^{\frac{1}{q}} \qquad \forall q > p.$$

As $\nabla\times\boldsymbol{h}_\infty \in \boldsymbol{L}^\infty(\Omega)$ and $\|\nabla\times\boldsymbol{h}_\infty\|_{\boldsymbol{L}^\infty(\Omega)} \leq 1$, we have that $\boldsymbol{h}_\infty$ belongs to the convex set $\mathbb{K}_\infty$.

Choosing, in (3.10), $\boldsymbol{\varphi} = \boldsymbol{v} - \boldsymbol{h}_n \in \mathbb{W}^\infty(\Omega)$ as test function, by monotonicity we have

$$\int_\Omega \nabla_n\times\boldsymbol{v}\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}_n) + \int_\Omega \boldsymbol{\delta}(x,\nabla\times\boldsymbol{v})\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}_n) \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h}_n) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h}_n)$$

and imposing that $\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} < 1$, letting $n \to \infty$, we obtain that $\boldsymbol{h}_\infty$ satisfies

$$\int_\Omega \boldsymbol{\delta}(x,\nabla\times\boldsymbol{v})\cdot\nabla\times(\boldsymbol{v}-\boldsymbol{h}_\infty) \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h}_\infty) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h}_\infty), \tag{3.15}$$

for any $\boldsymbol{v} \in \mathbb{K}_\infty$ such that $\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^\infty(\Omega)} < 1$ and, by density, also for any $\boldsymbol{v} \in \mathbb{K}_\infty$.

Choosing now, in (3.15), $\boldsymbol{v} = \lambda\boldsymbol{\varphi} + (1-\lambda)\boldsymbol{h}_\infty$, where $0 < \lambda \leq 1$ and $\boldsymbol{\varphi} \in \mathbb{K}_\infty$ is arbitrary, we easily see, after letting $\lambda \to 0$, that $\boldsymbol{h}_\infty$ satisfies (3.13). □

**Remark 3.2.** Notice that the above argument holds even in the case where $\boldsymbol{\delta} = 0$. In this degenerate case, we have shown the existence of $\boldsymbol{h}^0_\infty \in \mathbb{K}_\infty$ such that

$$\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{\varphi} - \boldsymbol{h}^0_\infty) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{\varphi} - \boldsymbol{h}^0_\infty) \leq 0 \qquad \forall \boldsymbol{\varphi} \in \mathbb{K}_\infty.$$

## 4. A stationary electromagnetic induction heating problem

### 4.1. *Weak formulation of the coupled system*

Considering the coupled problem (1.5)-(1.6) and using the notation (3.9) for the operator $p$-curl, and, similarly,

$$\nabla_q\theta = |\nabla\theta|^{q-2}\nabla\theta,$$

we introduce the weak formulation of the stationary electromagnetic induction heating problem.

To find $(\boldsymbol{h},\, \theta) \in \mathbb{W}^p(\Omega) \times W^{1,r}_0(\Omega)$ (for a certain $r : 1 \vee (q-1) \leq r \leq q$) such that

$$\int_\Omega \nu(\theta)\nabla_p\times\boldsymbol{h}\cdot\nabla\times\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{\varphi} \qquad \forall\boldsymbol{\varphi} \in \mathbb{W}^p(\Omega), \tag{4.1a}$$

$$\int_\Omega k(\theta)\nabla_q\theta\cdot\nabla\xi = \int_\Omega \nu(\theta)|\nabla\times\boldsymbol{h}|^p\xi \qquad \forall\xi \in W^{1,\infty}_0(\Omega). \tag{4.1b}$$

For technical reasons we shall restrict ourselves to the case $p > \frac{6}{5}$ and $q > \frac{5}{3}$.

Here we assume that $\nu, k : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions such that

$$0 < \nu_* \leq \nu(\theta) \leq \nu^* \text{ and } 0 < k_* \leq k(\theta) \leq k^* \quad \forall\theta \in \mathbb{R}. \tag{4.2}$$

**Theorem 4.1.** *Letting $p > \frac{6}{5}$ and $q > \frac{5}{3}$, $\nu, k : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy (4.2) and $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{p'}(\Gamma)$, the problem (4.1) has a solution with $r = q$ if $q > 3$ and $1 < r < \frac{3}{2}(q-1)$ if $\frac{5}{3} < q \leq 3$.*

To solve this problem, we consider a family of approximated problems by truncation.

Let us define the truncation operator as follows: given $M > 0$ and $v$ a real function,

$$\tau_M(v) = (v \wedge M) \vee (-M) = \begin{cases} -M & \text{if } v \leq -M, \\ v & \text{if } -M < v < M, \\ M & \text{if } v \geq M. \end{cases}$$

For $M > 0$ consider the approximated problem of (4.1) which consists of finding

$(\boldsymbol{h}_M, \theta_M) \in \mathbb{W}^p(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\int_\Omega \boldsymbol{\nu}(\theta_M)\nabla_p\times\boldsymbol{h}_M\cdot\nabla\times\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{\varphi} \qquad \forall\boldsymbol{\varphi}\in\mathbb{W}^p(\Omega), \tag{4.3a}$$

$$\int_\Omega k(\theta_M)\nabla_q\theta_M\cdot\nabla\xi = \int_\Omega \tau_M\big(\nu(\theta_M)|\nabla\times\boldsymbol{h}_M|^p\big)\xi \quad \forall\xi\in W_0^{1,q}(\Omega). \tag{4.3b}$$

**Proposition 4.1.** *For any $p > \frac{6}{5}$ and $q > 1$ the problem (4.3) has a solution.*

**Remark 4.1.** In fact, the function $\left|\tau_M(\nu(\theta_M)|\nabla\times\boldsymbol{h}_M|^p\right|$ is *a priori* bounded by $M$, and a well-known classical global estimate for $\theta_M$ (see Ref. 16, Section 10.5, for instance) yields

$$\|\theta_M\|_{L^\infty(\Omega)} \le C\,M,$$

where $C$ is a positive constant depending only on $\Omega$, $q$ and $k_*$. So, for the existence result of Proposition 4.1, the upper bounds $\nu^*$ and $k^*$ in the assumption (4.2) are not necessary.

**Remark 4.2.** More generally, we can consider a coupled problem for $(\boldsymbol{h},\theta) \in \mathbb{W}^p(\Omega)\times W_0^{1,r}(\Omega)$ (where, as before, $1\vee(q-1)\le r\le q$) of the form

$$\int_\Omega \boldsymbol{a}_p(x,\theta,\boldsymbol{h},\nabla\times\boldsymbol{h})\cdot\nabla\times\boldsymbol{\varphi} = \langle L,\boldsymbol{\varphi}\rangle \qquad \forall\,\boldsymbol{\varphi}\in\mathbb{W}^p(\Omega),$$

$$\int_\Omega \boldsymbol{b}_q(x,\theta,\boldsymbol{h},\nabla h)\cdot\nabla\xi = \int_\Omega \big(\boldsymbol{a}_p(x,\theta,\boldsymbol{h},\nabla\times\boldsymbol{h})\cdot\nabla\times\boldsymbol{h}\big)\xi \quad \forall\,\xi\in W_0^{1,\infty}(\Omega),$$

for a general $L\in\mathbb{W}^p(\Omega)'$ and quasi-linear Carathodory functions

$$\boldsymbol{a}_p,\boldsymbol{b}_q : \Omega\times\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

satisfying the structural conditions (2.2), respectively for $p > \frac{6}{5}$ and $q > \frac{5}{3}$, uniformly in the variables $(x,\theta,\boldsymbol{h})$.

**Remark 4.3.** The case $1 < q \le \frac{5}{3}$ can still be considered in the framework of entropy solutions introduced by Bnilan et al.[4] It consists in replacing (4.1b) by

$$\int_\Omega k(\theta)\nabla_q\theta\cdot\nabla\tau_s(\theta-\xi) = \int_\Omega \nu(\theta)|\nabla\times\boldsymbol{h}|^p\tau_s(\theta-\xi) \quad \forall\,\xi\in W_0^{1,\infty}(\Omega) \quad \forall\,s>0,$$

and the entropy solution is such that $\tau_s(\theta)\in W_0^{1,q}(\Omega)$, for every $s>0$. A continuous dependence result for entropy solutions (see Ref. 19) and their truncates allows to extend the existence result to this case.

### 4.2. *Existence of weak solutions*

**Proof of Proposition 4.1:** The proof will be done using the Schauder fixed point theorem.

Given $R > 0$, we consider $D_R = \{\gamma \in L^q(\Omega) : \|\gamma\|_{L^q(\Omega)} \leq R\}$ and, fixing $\gamma \in D_R$, we solve the auxiliary problem

$$\int_\Omega \boldsymbol{\nu}(\gamma)\nabla_p\times\boldsymbol{h}\cdot\nabla\times\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{\varphi} \quad \forall\boldsymbol{\varphi}\in\mathbb{W}^p(\Omega). \tag{4.4}$$

This problem is exactly the problem (2.5), for which we proved the existence of unique solution in the Proposition 2.1. Calling the solution $\boldsymbol{h}(\gamma)$ and using it as test function in (4.4) and recalling (4.2), there exists $C > 0$, depending only on $\nu_*$, $p$, $\Omega$, $\boldsymbol{f}$ and $\boldsymbol{g}$, such that

$$\|\nabla\times\boldsymbol{h}(\gamma)\|_{\boldsymbol{L}^p(\Omega)} \leq C. \tag{4.5}$$

Observe that, using the continuous dependence result of Theorem 3.1, we have that the function $S_1 : L^q(\Omega) \longrightarrow \mathbb{W}^p(\Omega)$ defined by $S_1(\gamma) = \boldsymbol{h}(\gamma)$ is continuous for the strong topologies.

Considering now $\gamma$ and $S_1(\gamma) = \boldsymbol{h}(\gamma)$ fixed, we solve the problem

$$\int_\Omega k(\gamma)\nabla_q\theta\cdot\nabla\xi = \int_\Omega \tau_M\big(\nu(\gamma)|\nabla\times\boldsymbol{h}(\gamma)|^p\big)\,\xi \quad \forall\xi\in W_0^{1,q}(\Omega). \tag{4.6}$$

This elliptic problem has a unique solution $\theta(\gamma) = \theta(\gamma, \boldsymbol{h}(\gamma)) \in W_0^{1,q}(\Omega)$ (see Ref. 20).

Using $\theta(\gamma)$ as a test function in (4.6), we get

$$k_*\int_\Omega |\nabla\theta(\gamma)|^q \leq \big\|\tau_M\big(\nu(\gamma)|\nabla\times\boldsymbol{h}(\gamma)|^p\big)\big\|_{L^{q'}(\Omega)}\,\|\theta(\gamma)\|_{L^q(\Omega)}$$

and applying the Poincar inequality,

$$\|\theta(\gamma)\|_{W_0^{1,q}(\Omega)} \leq C_M, \tag{4.7}$$

where $C_M > 0$ depends on $M$ and $\Omega$.

It is now easy to see that the function

$$\begin{array}{rcccl} S_2 : L^q(\Omega) & \longrightarrow & \mathbb{W}^p(\Omega) & \longrightarrow & W_0^{1,q}(\Omega), \\ \gamma & \longmapsto & \boldsymbol{h}(\gamma) & \longmapsto & \theta(\gamma,\boldsymbol{h}(\gamma)) \end{array}$$

is continuous. Indeed if $\{\gamma_n\}_n$ is a sequence in $L^q(\Omega)$ such that $\gamma_n \xrightarrow[n]{} \gamma$, then by Theorem 3.1, $|\nabla\times\boldsymbol{h}(\gamma_n)|^p \xrightarrow[n]{} |\nabla\times\boldsymbol{h}(\gamma)|^p$ in $L^1(\Omega)$.

Arguing as in the proof of Theorem 3.1 for the Dirichlet problem (4.3b), which satisfies the structural assumption (3.1) with $q > 1$, as in Remark 3.1, we easily obtain the estimate

$$\begin{aligned} \big\|\nabla\big(\theta(\gamma_n) - \theta(\gamma)\big)\big\|_{L^q(\Omega)}^{(q-1)\vee 1} \leq C\,\|k(\gamma) - k(\gamma_n)\|_{L^\infty(\Omega)} + \\ \big\|\tau_M\big(\nu(\gamma)|\nabla\times\boldsymbol{h}(\gamma_n)|^p\big) - \tau_M\big(\nu(\gamma)|\nabla\times\boldsymbol{h}(\gamma)|^p\big)\big\|_{L^{q'}(\Omega)}, \end{aligned} \tag{4.8}$$

so $\theta(\gamma_n) \xrightarrow[n]{} \theta(\gamma)$ in $W_0^{1,q}(\Omega)$ and the continuity of $S_2$ follows.

Recalling the compact imbedding of $W_0^{1,q}(\Omega)$ into $L^q(\Omega)$ the Schauder fixed point theorem in $L^q(\Omega)$ yields a solution to (4.3) with $\boldsymbol{h}_M = \boldsymbol{h}(\theta_M)$. □

**Proof of Theorem 4.1:** For $q > 3$, by the Sobolev imbedding $W_0^{1,q}(\Omega) \subset C^0(\bar{\Omega})$, repeating the proof of the Proposition 4.1 with (4.4) and (4.6) without the truncation $\tau_M$, we obtain the *a priori* estimate (4.5) and (4.7) with a constant $C' > 0$ depending only on $\Omega$, on $C$ of (4.5) and on the constants $k_*$ and $\nu^*$ in (4.2). Hence the same fixed point argument yields directly a solution $(\boldsymbol{h}(\theta), \theta)$ to (4.1).

For $\frac{5}{3} < q \leq 3$, we consider solutions $(\boldsymbol{h}_M, \theta_M)$ of (4.3) and we note that

$$\left\|\tau_M\big(\nu(\gamma)|\nabla\times\boldsymbol{h}_M|^p\big)\right\|_{L^1(\Omega)} \leq \|\nu(\gamma)|\nabla\times\boldsymbol{h}_M|^p\|_{L^1(\Omega)} \leq \nu^* \|\nabla\times\boldsymbol{h}_M\|^p_{L^p(\Omega)} \leq C^*, \tag{4.9}$$

where $C^* > 0$ is a constant independent of $M$.

By the estimates of Boccardo-Gallout,[6] we have for $1 \leq r < \frac{3}{2}(q-1)$ the *a priori* estimate

$$\|\theta_M\|_{W_0^{1,r}(\Omega)} \leq C_r,$$

where $C_r > 0$ is a constant also independent of $M$ but depending on $C^*$ of (4.9) and on the constants of (4.2).

Then, by compactness, at least for a subsequence $M \to \infty$ we may suppose

$$\boldsymbol{h}_M \longrightarrow \boldsymbol{h} \quad \text{in } \boldsymbol{L}^p(\Omega), \tag{4.10a}$$

$$\nabla\times\boldsymbol{h}_M \rightharpoonup \nabla\times\boldsymbol{h} \quad \text{in } \boldsymbol{L}^p(\Omega)\text{-weak}, \tag{4.10b}$$

$$\theta_M \longrightarrow \theta \quad \text{in } L^r(\Omega), \tag{4.10c}$$

$$\nabla\theta_M \rightharpoonup \nabla\theta \quad \text{in } \boldsymbol{L}^r(\Omega)\text{-weak}. \tag{4.10d}$$

and, using the assumption (4.2), also

$$\nu(\theta_M) \longrightarrow \nu(\theta) \quad \text{in } L^s(\Omega),\ \forall\, 1 < s < \infty \tag{4.10f}$$

$$\nu(\theta_M)\nabla_p\times\boldsymbol{h}_M \rightharpoonup \boldsymbol{\lambda} \quad \text{in } \boldsymbol{L}^{p'}(\Omega)\text{-weak}, \tag{4.10g}$$

since $\|\nu(\theta_M)\nabla_p\times\boldsymbol{h}_M\|_{\boldsymbol{L}^{p'}(\Omega)} \leq \nu^* \|\nabla\times\boldsymbol{h}_M\|^{p-1}_{\boldsymbol{L}^p(\Omega)} \leq C''$, where $C''$ does not depend on $M$ by (4.5).

From (4.3a) and (4.10g) we see that $\boldsymbol{h}$ solves (4.1a) if we show

$$\int_\Omega \boldsymbol{\lambda}\cdot\nabla\times\boldsymbol{\varphi} = \int_\Omega \nu(\theta)\nabla_p\times\boldsymbol{h}\cdot\nabla\times\boldsymbol{\varphi} \quad \forall\boldsymbol{\varphi} \in \mathbb{W}^p(\Omega), \tag{4.11}$$

which can be done by an adaptation of Minty's lemma (see Ref. 20).

Indeed, on one hand, letting $M \to \infty$ in (4.3a), we obtain first

$$\int_\Omega \boldsymbol{\lambda}\cdot\nabla\times\boldsymbol{\varphi} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi} + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{\varphi} \quad \forall\boldsymbol{\varphi} \in \mathbb{W}^p(\Omega)$$

and, noting that $\boldsymbol{h} \in \mathbb{W}^p(\Omega)$, also

$$\lim_M \int_\Omega \nu(\theta_M)\nabla_p\times\boldsymbol{h}_M\cdot\nabla\times\boldsymbol{h}_M = \lim_M \left(\int_\Omega \boldsymbol{f}\cdot\boldsymbol{h}_M + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{h}_M\right) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{h} + \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{h} = \int_\Omega \boldsymbol{\lambda}\cdot\nabla\times\boldsymbol{h}. \tag{4.12}$$

By monotonicity, we have

$$\int_\Omega \nu(\theta_M)\nabla_p\times\boldsymbol{h}_M\cdot\nabla\times(\boldsymbol{h}_M-\boldsymbol{v})-\int_\Omega \nu(\theta)\nabla_p\times\boldsymbol{v}\cdot\nabla\times(\boldsymbol{h}_M-\boldsymbol{v})\geq \int_\Omega \big(\nu(\theta_M)-\nu(\theta)\big)\nabla_p\times\boldsymbol{v}\cdot\nabla\times(\boldsymbol{h}_M-\boldsymbol{v})$$

and, letting $M\to\infty$, using (4.12), we obtain

$$\int_\Omega \big(\boldsymbol{\lambda}-\nu(\theta)\nabla_p\times\boldsymbol{v}\big)\cdot\nabla\times(\boldsymbol{h}-\boldsymbol{v})\geq 0\quad\forall\boldsymbol{v}\in\mathbb{W}^p(\Omega).\tag{4.13}$$

Choosing $\boldsymbol{v}=\boldsymbol{h}-\alpha\,\boldsymbol{\varphi}$ with $\boldsymbol{\varphi}\in\mathbb{W}^p(\Omega)$, dividing by $\alpha>0$ and letting $\alpha\to 0$, we obtain from (4.13)

$$\int_\Omega \big(\boldsymbol{\lambda}-\nu(\theta)\nabla_p\times\boldsymbol{h}\big)\cdot\nabla\times\boldsymbol{\varphi}\geq 0\quad\forall\boldsymbol{\varphi}\in\mathbb{W}^p(\Omega),$$

which implies (4.11).

Taking $\boldsymbol{\varphi}=\boldsymbol{h}_M-\boldsymbol{h}$ in (4.1a) and (4.3a) we obtain

$$\int_\Omega \nu(\theta_M)\left(\nabla_p\times\boldsymbol{h}_M-\nabla_p\times\boldsymbol{h}\right)\cdot\nabla\times(\boldsymbol{h}_M-\boldsymbol{h})= \int_\Omega \left(\nu(\theta)-\nu(\theta_M)\right)\nabla_p\times\boldsymbol{h}\cdot\nabla\times(\boldsymbol{h}_M-\boldsymbol{h}).\tag{4.14}$$

By Lebesgue theorem and (4.10f) we have $\big(\nu(\theta)-\nu(\theta_M)\big)\nabla_p\times\boldsymbol{h}\to 0$ in $\boldsymbol{L}^{p'}(\Omega)$, so the right-hand side of (4.14) tends to zero. The assumptions (4.2) imply that

$$\nabla\times\boldsymbol{h}_M \xrightarrow[M]{} \nabla\times\boldsymbol{h}\quad\text{in }\boldsymbol{L}^p(\Omega),$$

using, when $p<2$, arguments similar to (3.6). By (4.10f), we have

$$\nu(\theta_M)|\nabla\times\boldsymbol{h}_M|^p \xrightarrow[M]{} \nu(\theta)|\nabla\times\boldsymbol{h}|^p\quad\text{in }L^1(\Omega)$$

and, consequently, also

$$\tau_M\big(\nu(\theta_M)|\nabla\times\boldsymbol{h}_M|^p\big) \xrightarrow[M]{} \nu(\theta)|\nabla\times\boldsymbol{h}|^p\quad\text{in }L^1(\Omega).$$

Finally, in order to show that $\theta$ also solves (4.1b) we apply Lemma 1 of Boccardo-Gallout,[7] which implies that $\{\theta_M\}_M$ is compact in $W_0^{1,r}(\Omega)$, for any $1\leq r<\frac{3}{2}(q-1)$ and therefore we have

$$k(\theta_M)\nabla_q\theta_M \xrightarrow[M]{} k(\theta)\nabla_q\theta\quad\text{in }L^s(\Omega)\quad\forall\, 1\leq s<\tfrac{3}{2}.$$

□

## 5. A stationary magnetization of a superconductor

### 5.1. *The quasi-variational inequality*

Similarly to the quasi-variational inequality (1.10) considered in the introduction, and substituting the operator $p$-curl by the more general operator $\boldsymbol{a}$, we are lead to the following quasi-variational inequality,

$$\begin{cases} \boldsymbol{h} \in \mathbb{K}_{\boldsymbol{h}} = \big\{ \boldsymbol{v} \in \mathbb{W}^p(\Omega) : |\nabla\times\boldsymbol{v}| \leq F(|\boldsymbol{h}|) \text{ a.e in } \Omega \big\} \\ \displaystyle\int_\Omega \boldsymbol{a}(x, \nabla\times\boldsymbol{h}) \cdot \nabla\times(\boldsymbol{v}-\boldsymbol{h}) \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h}) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h}) \quad \forall\, \boldsymbol{v} \in \mathbb{K}_{\boldsymbol{h}}, \end{cases} \tag{5.1}$$

where the operator $\boldsymbol{a}$ satisfies the assumptions (2.2).

**Theorem 5.1.** *Suppose that* $p > \frac{6}{5}$, $\boldsymbol{f} \in \boldsymbol{L}^{p'}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{L}^{p'}(\Gamma)$ *and* $F : [0,\infty) \longrightarrow \mathbb{R}$ *is continuous and there exists* $\nu > 0$ *such that*

$$F(s) \geq \nu > 0 \quad \forall\, s \geq 0. \tag{5.2}$$

*If* $\frac{6}{5} < p \leq 3$ *suppose, in addition, that there exists positive constants* $c_0$ *and* $c_1$ *such that*

$$F(s) \leq c_0 + c_1 s^\alpha \quad \forall\, s \geq 0, \tag{5.3}$$

*where* $\alpha \geq 0$ *if* $p = 3$ *and* $0 \leq \alpha \leq \frac{p}{3-p}$ *if* $\frac{6}{5} < p < 3$.

*Then the quasi-variational inequality* (5.1) *has a solution.*

### 5.2. *A continuous dependence result*

We consider first the following variational inequality, defined for a given nonnegative function $\varphi \in L^\infty(\Omega)$,

$$\begin{cases} \boldsymbol{h} \in \mathbb{K}_\varphi = \big\{ \boldsymbol{v} \in \mathbb{W}^p(\Omega) : |\nabla\times\boldsymbol{v}| \leq F(\varphi) \text{ a.e in } \Omega \big\} \\ \displaystyle\int_\Omega \boldsymbol{a}(x, \nabla\times\boldsymbol{h}) \cdot \nabla\times(\boldsymbol{v}-\boldsymbol{h}) \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{h}) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{h}) \ \forall\, \boldsymbol{v} \in \mathbb{K}_\varphi. \end{cases} \tag{5.4}$$

**Lemma 5.1.** *Suppose we have a sequence* $\{\varphi_n\}_n$ *in* $L^\infty(\Omega)$ *such that* $\varphi_n \underset{n}{\longrightarrow} \varphi$ *in* $L^\infty(\Omega)$. *Let* $F$ *satisfy the assumption* (5.2) *and the operator* $\boldsymbol{a}$ *verify* (2.2). *If* $\boldsymbol{h}_n$ *and* $\boldsymbol{h}$ *denote the solution of the variational inequality* (5.4) *for given* $\varphi_n$ *and* $\varphi$, *respectively, then*

$$\boldsymbol{h}_n \underset{n}{\longrightarrow} \boldsymbol{h} \quad \text{in } \mathbb{W}^p(\Omega). \tag{5.5}$$

**Proof.** We follow arguments similar to Ref. 2 by proving the Mosco convergence of the family of convex sets $\mathbb{K}_{\varphi_n}$ to $\mathbb{K}_\varphi$.

Given a sequence $\{\boldsymbol{v}_n\}_n$ belonging to $\mathbb{K}_{\varphi_n}$, we need to prove that if $\boldsymbol{v}_n \longrightarrow \boldsymbol{v}$ in $\mathbb{W}^p(\Omega)$ then $\boldsymbol{v} \in \mathbb{K}_\varphi$. It is enough to observe that, for $\boldsymbol{v}_n$ belonging to $\mathbb{K}_{\varphi_n}$ we have $|\nabla\times\boldsymbol{v}_n| \leq F(\varphi_n)$ and, since for any measurable $\omega \subset \Omega$,

$$\int_\omega |\nabla\times\boldsymbol{v}| \leq \liminf_n \int_\omega |\nabla\times\boldsymbol{v}_n| \leq \liminf_n \int_\omega F(\varphi_n) = \int_\omega F(\varphi),$$

we get $|\nabla\times\boldsymbol{v}| \leq F(\varphi)$ a.e. in $\Omega$.

Given $\boldsymbol{v} \in \mathbb{K}_\varphi$ we need to construct a sequence $\boldsymbol{v}_n \in \mathbb{K}_{\varphi_n}$ such that $\boldsymbol{v}_n \xrightarrow[n]{} \boldsymbol{v}$ in $\mathbb{W}^p(\Omega)$. Defining $\lambda_n = \|F(\varphi_n) - F(\varphi)\|_{L^\infty(\Omega)}$, we have $\lambda_n \xrightarrow[n]{} 0$. If we define $\boldsymbol{v}_n = \frac{1}{\mu_n}\boldsymbol{v}$, for $\mu_n = 1 + \frac{\lambda_n}{\nu}$ then $\boldsymbol{v}_n \in \mathbb{W}^p(\Omega)$ and

$$|\nabla\times\boldsymbol{v}_n| = \frac{1}{\mu_n}|\nabla\times\boldsymbol{v}| \leq \frac{1}{\mu_n}F(\varphi) \leq F(\varphi_n)$$

since

$$\mu_n = 1 + \frac{\|F(\varphi_n) - F(\varphi)\|_\infty}{\nu} \geq 1 + \frac{F(\varphi) - F(\varphi_n)}{F(\varphi_n)} = \frac{F(\varphi)}{F(\varphi_n)},$$

which means that $\boldsymbol{v}_n \in \mathbb{K}_{\varphi_n}$. Besides that

$$\|\boldsymbol{v}_n - \boldsymbol{v}\|^p_{\mathbb{W}^p(\Omega)} = \int_\Omega |\nabla\times(\boldsymbol{v}_n - \boldsymbol{v})|^p = \int_\Omega \Big(1 - \frac{1}{\mu_n}\Big)^p |\nabla\times\boldsymbol{v}|^p = \tfrac{\lambda_n}{\nu}\|\boldsymbol{v}\|^p_{\mathbb{W}^p(\Omega)} \xrightarrow[n]{} 0$$

and so the Mosco convergence is proved. By a well known result of Mosco (see Ref. 25), we have (5.5). □

### 5.3. *Existence of solution of the quasi-variational inequality*

**Proof of Theorem 5.1:** Following Ref. 17, we use a fixed point argument to prove this theorem.

For a given $\varphi \in C(\bar{\Omega})$, we denote by $\boldsymbol{h}_\varphi$ the unique solution of the variational inequality (5.4) with convex $\mathbb{K}_\varphi$ and we define

$$\begin{aligned} S : C(\bar{\Omega}) &\longrightarrow \mathbb{W}^p(\Omega). \\ \varphi &\longmapsto \boldsymbol{h}_\varphi \end{aligned}$$

The continuity of $S$ is a consequence of Lemma 5.1.

For $p > 3$, since $\mathbb{W}^p(\Omega)$ is a subspace of $\boldsymbol{W}^{1,p}(\Omega)$ and this last space is compactly imbedded in $\boldsymbol{C}(\bar{\Omega})$, we have

$$\begin{array}{ccccccc} \tilde{S} : C(\bar{\Omega}) & \rightarrow & \mathbb{W}^p(\Omega) & \hookrightarrow & \boldsymbol{C}(\bar{\Omega}) & \rightarrow & C(\bar{\Omega}) \\ \varphi & \mapsto & \boldsymbol{h}_\varphi & \mapsto & \boldsymbol{h}_\varphi & \mapsto & |\boldsymbol{h}_\varphi| \end{array} \tag{5.6}$$

is a continuous and compact mapping.

Taking $\boldsymbol{v} = 0$ in (5.4), we easily conclude that

$$\|\boldsymbol{h}_\varphi\|_{\mathbb{W}^p(\Omega)} \leq C, \tag{5.7}$$

where $C > 0$ is independent of the $\varphi$ but depends on $a^*$, $\Omega$, $\boldsymbol{f}$ and $\boldsymbol{g}$. Therefore, there exists some $R > 0$, such that we have $\tilde{S}\left(C(\bar{\Omega})\right) \subseteq D_R$, for $D_R = \left\{\varphi \in C(\bar{\Omega}) : \|\varphi\|_{C(\bar{\Omega})} \leq R\right\}$. Applying the Schauder fixed point theorem, we get that $\tilde{S}$ has a fixed point in $D_R$, which solves (5.1).

To prove the case $\frac{6}{5} < p \leq 3$, we apply Leray-Schauder principle (see e.g. Ref. 16, p. 280). We start noticing that, given $\varphi \in C(\bar{\Omega})$, the solution $\boldsymbol{h}_\varphi$ of (5.4) belongs to $\mathbb{W}^\infty(\Omega)$.

Besides that, given $\psi \in C(\bar{\Omega})$ and for any $r > 3$ we have

$$\|\boldsymbol{h}_\varphi - \boldsymbol{h}_\psi\|^r_{\mathbb{W}^r(\Omega)} = \int_\Omega |\nabla\times(\boldsymbol{h}_\varphi - \boldsymbol{h}_\psi)|^r \leq$$
$$2^{r-p-1} \int_\Omega \left(|F(\varphi)|^{r-p} + |F(\psi)|^{r-p}\right) \left|\nabla\times(\boldsymbol{h}_\varphi - \boldsymbol{h}_\psi)\right|^p. \quad (5.8)$$

So, using Lemma 5.1, the function $\tilde{S}$, defined as in (5.6), with $\mathbb{W}^p(\Omega)$ replaced by $\mathbb{W}^r(\Omega)$ is continuous and compact. To prove that $\tilde{S}$ has a fixed point it is enough to prove that the set

$$\mathcal{A} = \left\{\varphi \in C(\bar{\Omega}) : \varphi = \lambda\tilde{S}(\varphi) \text{ for some } \lambda \in [0,1]\right\}$$

is bounded independently of $\lambda$.

Given $\varphi \in \mathcal{A}$, we have $\varphi = \lambda\tilde{S}(\varphi) = \lambda|\boldsymbol{h}_\varphi|$, for some $\lambda \in [0,1]$. So

$$\|\varphi\|^r_{C(\bar{\Omega})} = \lambda^r \|\,|\boldsymbol{h}_\varphi|\,\|^r_{C(\bar{\Omega})} \leq c\,\|\boldsymbol{h}_\varphi\|^r_{\mathbb{W}^r(\Omega)} = c\int_\Omega |\nabla\times\boldsymbol{h}_\varphi|^r \leq$$
$$c\int_\Omega |F(\varphi)|^r \leq c\int_\Omega \left(c_0 + c_1|\varphi|^\alpha\right)^r = \tilde{c}_0 + \tilde{c}_1|\lambda|^{r\alpha}\int_\Omega |\boldsymbol{h}_\varphi|^{r\alpha} \leq$$
$$\tilde{c}_0 + c_2\|\boldsymbol{h}_\varphi\|^{r\alpha}_{\boldsymbol{W}^{1,p}(\Omega)} \leq \tilde{c}_0 + c_3\left(\int_\Omega |\nabla\times\boldsymbol{h}_\varphi|^p\right)^{\frac{r\alpha}{p}},$$

using the Sobolev inclusion $W^{1,p}(\Omega) \subset L^{r\alpha}(\Omega)$, which is verified by the assumption (5.3) on $\alpha$. The conclusion of the boundedeness of $\mathcal{A}$ follows then directly from (5.7). □

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