Weak sectional category

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Abstract

Based on a Whitehead-type characterization of the sectional category we develop the notion of weak sectional category. This is a new lower bound of the sectional category, which is inspired by the notion of weak category in the sense of Berstein-Hilton. We establish several properties and inequalities, including the fact that the weak sectional category is a better lower bound for the sectional category than the classical one given by the nilpotency of the kernel of the induced map in cohomology. Finally, we apply our results in the study of the topological complexity in the sense of Farber.

2000 *Mathematics Subject Classification* : 55M30.
Keywords : Sectional category, Topological complexity.

Introduction.

For a fibration $p : E \to B$, the sectional category $\text{secat}\,(p)$ is defined as the least integer k such that B admits a cover constituted by $k+1$ open subsets, on each of which p has a section. Requiring homotopy section instead of section permits to extend the notion of sectional category to any map. This is a variant of Lusternik-Schnirelmann category (or L-S category, for short) and also a generalization, since $\text{secat}\,(p) = \text{cat}\,(B)$ when $E \simeq *$. The sectional category has been introduced for fibrations by A. Schwarz [13] in the 1960's, under the name *genus* (then it was renamed by I.M. James [11]). It has turned out to be a useful tool not only in questions concerning the classification of bundles, the embedding problem (see [13] for both applications) or the complexity of the root-finding problem for algebraic equations [15], but also, more recently, in the study of the motion planning problem in robotics [7], [8].

This article has been submitted for publication.
This work has been supported by the Ministerio de Educación y Ciencia grant MTM2006-06317, the Portuguese Foundation for Science and Technology (FCT/POCI 2010) and FEDER.

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In the development of L-S category, Ganea's and Whitehead's characterizations have played a very important role. These characterizations are not only easier to handle in the framework of homotopy theory than the open set definition but they also permit to obtain many approximations of L-S category. Since one of the most successful aproximations of L-S category is the weak category in the sense of Berstein-Hilton [2], our aim in this paper is to develop an analogous lower bound for sectional category, the *weak sectional category*. Taking into account that the classical weak category is based on Whitehead's characterization we need a Whitehead-type formulation of sectional category. Such a characterization has been obtained by A. Fassò [9]. More precisely, for a cofibration $f : A \to X$ and an integer $n \geq 0$, she defined the n-fatwedge of f as the subspace of X^{n+1} given by $T^n(f) = \{(x_0, ..., x_n) \in X^{n+1} \,|\, x_i \in A, \text{ for some } i\}$ and she proves that, for a map $p : E \to B$ with associated cofibration $\hat{p} : E \to \hat{B}$, $\operatorname{secat}(p) \leq n$ if and only if the diagonal map $\Delta_{n+1} : \hat{B} \to \hat{B}^{n+1}$ lifts, up to homotopy, in the n-fatwegde of the cofibration $\hat{p}$:

$$\begin{array}{ccc} & & T^n(\hat{p}) \\ & \nearrow & \downarrow \\ \hat{B} & \xrightarrow[\Delta_{n+1}]{} & \hat{B}^{n+1} \end{array}$$

In section 1 we use a notion of fatwedge for any map which generalizes Fassò's fatwedge and recover her result. Namely, for any map $p : E \to B$, we construct what we call the n-sectional fatwedge of p, $\kappa_n : T^n(p) \to B^{n+1}$, and prove that $\operatorname{secat}(p) \leq n$ if and only if $\Delta_{n+1} : B \to B^{n+1}$ lifts, up to homotopy, in $T^n(p)$.

The notion of weak sectional category comes naturally. Considering C_{κ_n} the homotopy cofibre of the n-sectional fat wedge of p, $\kappa_n : T^n(p) \to B^{n+1}$, and considering $l_n : B^{n+1} \to C_{\kappa_n}$ the induced map, the weak sectional category of p, $\operatorname{wsecat}(p)$, is the least integer n (or ∞) such that the composition

$$B \xrightarrow{\Delta_{n+1}} B^{n+1} \xrightarrow{l_n} C_{\kappa_n}$$

is homotopically trivial. Section 2 is fully devoted to the study of this new lower bound of sectional category. This study can be summarized in the following theorem (see section 2 for more details)

Theorem. Let $p : E \to B$ be a map and C_p be its homotopy cofibre. Then

(a) $\operatorname{wsecat}(p) \leq \operatorname{wcat}(B)$

(b) $\operatorname{wsecat}(p) \leq \operatorname{wcat}(C_p)$

(c) $\operatorname{wsecat}(p) \geq \operatorname{wcat}(C_p) - 1$

(d) $\operatorname{wsecat}(p) \geq \operatorname{nil} \ker p^*$ (for any commutative ring π)

(e) If the map $p : E \to B$ admits a homotopy retraction, then

$$\mathrm{wsecat}\,(p) = \mathrm{wcat}\,(C_p) \text{ and } \mathrm{nil}\ker p^* = \mathrm{cuplength}\,(C_p).$$

Here $\mathrm{nil}\ker p^*$ denotes the nilpotency of the kernel of the morphism p^* which is induced by p in cohomology. This is a classical lower bound for sectional category [13]. By inequality (d) above, weak sectional category turns out to be a better lower bound than $\mathrm{nil}\ker p^*$ and we will see that actually the inequality can be strict. In Section 2 we also give sufficient conditions to assure that $\mathrm{wsecat}\,(p) = \mathrm{secat}\,(p)$.

On the other hand, the motion planning problem is an important research area in robotics in which topological methods can be applied. The motion planning problem of the configuration space X (associated to a mechanical system) consists of constructing a program or a devise, which takes pairs of configurations $(a, b) \in X \times X$ as an input and produces as an output a continuous path ω in X such that $\omega(0) = a$ and $\omega(1) = b$. Studying such a research area M. Farber introduced in [7], [8] the notion of topological complexity, which is a numerical homotopical invariant of the space X. It is defined as the sectional category of the evaluation fibration $\pi_X : X^I \to X \times X$, $\pi_X(\alpha) = (\alpha(0), \alpha(1))$. In the last section of this paper we study what we call *weak topological complexity*, $\mathrm{wTC}\,(X)$, which is nothing else but the weak sectional category of the evaluation fibration π_X. This is a new lower bound of the topological complexity. Specializing the results given in section 2 we obtain the corresponding ones, summarized as

Theorem. Let X be any space. Then

(a) $\mathrm{wTC}\,(X) \leq \mathrm{wcat}\,(X \times X)$

(b) $\mathrm{wTC}\,(X) \geq \mathrm{nil}\ker\,\cup$

(c) $\mathrm{wTC}\,(X) = \mathrm{wcat}\,(C_{\Delta_X})$ (and $\mathrm{nil}\ker\,\cup = \mathrm{cuplength}\,(C_{\Delta_X})$).

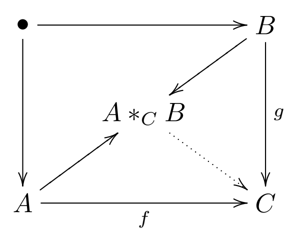
We also give sufficient conditions to assure that $\mathrm{wTC}\,(X) = \mathrm{TC}\,(X)$. Finally we give two concrete computations of wTC. The first one consists of the explicit determination of the homotopy cofibre of the diagonal map $\Delta_{S^n} : S^n \to S^n \times S^n$ of a sphere. This fact permits us to recover the result by Farber that the topological complexity of an odd dimensional sphere is 1 while that of an even dimensional sphere is 2. The second computation has the aim to prove, through an example, that wTC is, in general, a better lower bound for the topological complexity than $\mathrm{nil}\ker\,\cup$.

Throughout this paper we work with homotopy commutative diagrams, homotopy pullbacks, homotopy pushouts and joins as well as some of their more important properties. We assume that the reader is familiarized with this framework; nevertheless at the very beginning of Section 1 we recall the notion of join, the Join Theorem and the Prism Lemma. For more details we refer the reader to

[12], [6] and [3]. We also point out that the category in which we shall work is the category of well-pointed compactly generated Hausdorff spaces. Therefore all categorical constructions are carried out in this category.

1 Joins and characterizations of sectional category.

The main goal of this section is to give some important notions and results that will be needed in the rest of the paper. We begin by recalling the notion of join of two maps. Given any pair of maps $A \xrightarrow{f} C \xleftarrow{g} B$, the *join of* f *and* g, $A *_C B$, is the homotopy pushout of the homotopy pullback of f and g



being the dotted arrow the corresponding co-whisker map, induced by the weak universal property of homotopy pushouts.

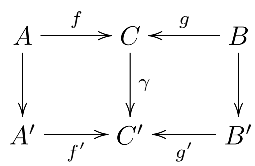
Notice that the map $A *_C B \to C$ is only defined up to weak equivalence. Any map constructed in such a way is weakly equivalent to the canonical co-whisker map obtained by considering first the standard homotopy pullback of f and g and then the standard homotopy pushout of the projections on A and B. In order to give another example of a concrete construction of the join, suppose that f (or g) is a fibration. Then the honest pullback of f and g is actually a homotopy pullback and taking the standard homotopy pushout of the projections $A \times_C B \to A$ and $A \times_C B \to B$ we obtain a representative of the map $A *_C B \to C$ of the form:

$$\begin{array}{rcl} A *_C B = A \amalg A \times_C B \times [0,1] \amalg B/\sim & \to & C \\ \langle a,b,t \rangle & \mapsto & f(a) = g(b) \end{array}$$

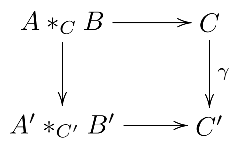
where $\sim$ is given by $(a,b,t) \sim a$ if $t = 0$ and $(a,b,t) \sim b$ if $t = 1$.

In [6], Doeraene established the following result, called the “Join Theorem”.

Theorem 1 (**Join Theorem**). *Consider the homotopy commutative diagram in* *which the squares are homotopy pullbacks*



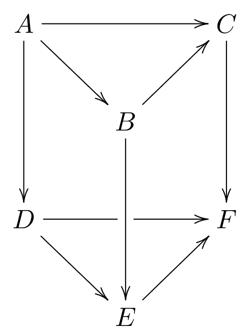
Then there is a homotopy pullback



where the horizontal arrows are the corresponding induced co-whisker maps.

We also mention the following classical result that we will use in the sequel, as well as its dual version for homotopy pushouts.

Proposition 2 (**Prism Lemma**). *Given any homotopy commutative diagram,*



where the square $BCFE$ is a homotopy pullback. Then, $ACFD$ is a homotopy pullback if and only $ABED$ is a homotopy pullback.

The basic notion of interest in this work is the sectional category:

Definition 3. *The sectional category of a map $p : E \to B$, $\mathrm{secat}(p)$, is the least integer n (or ∞) such that B can be covered by $n+1$ open subsets, on each of which p admits a homotopy section.*

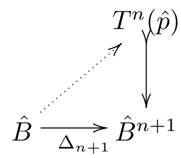
There is a classical Ganea-type characterization of this homotopy numerical invariant which is as follows. Given any map $p : E \to B$ we may consider the join map $E *_B E \to B$ resulting from the join of p with p. This construction can be done again, considering the join of this map with p. In this way we inductively obtain $j_p^n : *_B^n E \to B$ as the join of j_p^{n-1} and p (we set $j_p^0 = p$, $*_B^0 E = E$). The characterization of secat is then given by the following classical result, see for instance [11].

Theorem 4. *Let $p : E \to B$ be a map. If B is a paracompact space, then* secat $(p) \leq n$ *if and only if j_p^n admits a homotopy section.*

We remark that there is an improvement of the above theorem in [10], in which B is required to be just a normal space.

As said in the introduction, A. Fassò gave in [9] the following Whitehead-type characterization of sectional category:

Theorem 5. *Let $p : E \to B$ be a map and let $\hat{p} : E \to \hat{B}$ be the associated cofibration of p. Then* $\mathrm{secat}(p) \leq n$ *if and only if the diagonal map $\Delta_{n+1} : \hat{B} \to \hat{B}^{n+1}$ lifts, up to homotopy, in the n-fatwegde $T^n(\hat{p}) = \{(b_0, ..., b_n) \in \hat{B}^{n+1} \,|\, b_i \in E,$ for some $i\}$ of the cofibration $\hat{p}$:*



We give an alternative proof of this result based on a notion of fatwedge defined for any map and called *sectional fatwedge*. We note that Doeraene [5] used this construction for different purposes as an auxiliary notion under the name of relative fatwedge.

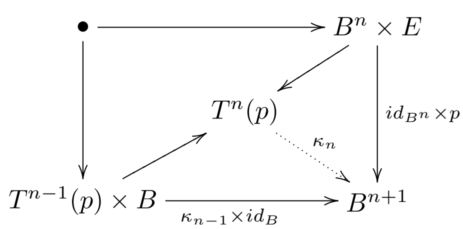
Definition 6. The *n-sectional fatwedge* of any map $p : E \to B$

$$\kappa_n : T^n(p) \to B^{n+1}$$

is defined inductively as follows:

(i) For $n = 0$ we set $\kappa_0 = p : E \to B$

(ii) Having defined $\kappa_{n-1} : T^{n-1}(p) \to B^n$ then κ_n is obtained by considering the join of $\kappa_{n-1} \times id_B$ and $id_{B^n} \times p$:



We will use the following lemma. Its proof is straightforward and therefore is omitted.

Lemma 7. Consider $f : X \to Y$ any map and Z any space. Then, the following square, where the horizontal arrows are the obvious projections, is a pullback and a homotopy pullback

$$\begin{array}{ccc} X \times Z & \xrightarrow{pr} & X \\ {\scriptstyle f\times id_Z}\downarrow & & \downarrow{\scriptstyle f} \\ Y \times Z & \xrightarrow[pr]{} & Y. \end{array}$$

Theorem 8. *Let $p : E \to B$ be any map and $n \geq 0$ any nonnegative integer. Then there is a homotopy pullback*

$$\begin{array}{ccc} *^n_B E & \xrightarrow{\widetilde{\Delta}_{n+1}} & T^n(p) \\ {\scriptstyle j^n_p}\downarrow & & \downarrow{\scriptstyle \kappa_n} \\ B & \xrightarrow[\Delta_{n+1}]{} & B^{n+1} \end{array}$$

Proof. We proceed by induction on the integer $n \geq 0$. For $n = 0$ the result is obviously true. Now suppose the result is true for $n-1$. Considering the Prism Lemma, the above lemma and the induction hypothesis to the following homotopy commutative diagram

$$\begin{array}{ccccc} *^{n-1}_B E & \xrightarrow{(\widetilde{\Delta}_n, j^{n-1}_p)} & T^{n-1}(p) \times B & \xrightarrow{pr} & T^{n-1}(p) \\ {\scriptstyle j^{n-1}_p}\downarrow & & \downarrow{\scriptstyle \kappa_{n-1}\times id_B} & & \downarrow{\scriptstyle \kappa_{n-1}} \\ B & \xrightarrow[\Delta_{n+1}]{} & B^{n+1} & \xrightarrow[pr]{} & B^n \end{array}$$

we have that the left square is a homotopy pullback. Now applying a similar argument to the homotopy commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{(p,\ldots,p,id_E)} & B^n \times E & \xrightarrow{pr} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle id_{B^n}\times p} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\Delta_{n+1}]{} & B^{n+1} & \xrightarrow[pr]{} & B \end{array}$$

we have that the left square is also a homotopy pullback. Summarizing, we have obtained a homotopy commutative diagram in which the squares are homotopy pullbacks

$$\begin{array}{ccccc} E & \xrightarrow{p} & B & \xleftarrow{j^{n-1}_p} & *^{n-1}_B E \\ {\scriptstyle (p,\ldots,p,id_E)}\downarrow & & \downarrow{\scriptstyle \Delta_{n+1}} & & \downarrow{\scriptstyle (\widetilde{\Delta}_n, j^{n-1}_p)} \\ B^n \times E & \xrightarrow[id_{B^n}\times p]{} & B^{n+1} & \xleftarrow[\kappa_{n-1}\times id_B]{} & T^{n-1}(p) \times B \end{array}$$

Applying the Join Theorem to this diagram we conclude the proof. □

Corollary 9. Let $p : E \to B$ be any map, where B is a normal space. Then one has $\mathrm{secat}\,(p) \leq n$ if and only if there is, up to homotopy, a lift of the $(n+1)$-diagonal map

$$\begin{array}{ccc} & & T^n(p) \\ & \nearrow & \downarrow{\scriptstyle \kappa_n} \\ B & \xrightarrow[\Delta_{n+1}]{} & B^{n+1} \end{array}$$

Now we will see that $T^n(p)$ coincides with Fassò's fatwedge when p is a cofibration.

Lemma 10. *Let $p : E \to B$ be any map. Then the following square is a homotopy pullback*

$$\begin{array}{ccc} T^{n-1}(p) \times E & \xrightarrow{\kappa_{n-1} \times id_E} & B^n \times E \\ {\scriptstyle id \times p}\downarrow & & \downarrow{\scriptstyle id_{B^n} \times p} \\ T^{n-1}(p) \times B & \xrightarrow[\kappa_{n-1} \times id_B]{} & B^{n+1} \end{array}$$

Proof. Take the standard homotopy pullback of $\kappa_{n-1} \times id_B$ and $id_{B^n} \times p$, that is

$$\begin{aligned} L \quad = \quad & \{(a, x, \gamma, b, y) \in T^{n-1}(p) \times B \times (B^{n+1})^I \times B^n \times E \text{ such that} \\ & \qquad \gamma(0) = (\kappa_{n-1}(a), x),\ \gamma(1) = (b, p(y))\} \end{aligned}$$

Consider the maps

$$\xi : L \to T^{n-1}(p) \times E \ \text{ and } \ \pi : T^{n-1}(p) \times E \to L$$

given by $\xi(a, x, \gamma, b, y) = (a, y)$ and $\pi(a, y) = (a, p(y), \varepsilon_{(\kappa_{n-1}(a), p(y))}, \kappa_{n-1}(a), y)$, where $\varepsilon_{(\kappa_{n-1}(a), p(y))}$ denotes the constant path on $(\kappa_{n-1}(a), p(y))$. Then $\xi\pi = id$ and $\pi\xi \simeq id$ through the homotopy $F : L \times I \to L$ defined as

$$F(a, x, \gamma, b, y; t) = (a, \gamma_2(t), \delta(t), \gamma_1(1 - t), y)$$

where $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 : I \to B^n$, $\gamma_2 : I \to B$ and

$$\delta(t)(s) = (\gamma_1(s(1 - t)), \gamma_2((1 - s)t + s))$$

This proves that π is a homotopy equivalence and therefore the statement of the lemma. □

Taking into account this fact we have that $T^n(p)$ may be directly obtained as the homotopy pushout of $\kappa_{n-1} \times id_E$ and $id \times p$. In particular when p is a cofibration we can consider the honest pushout. Using induction we have:

Corollary 11. *Let $p : E \rightarrowtail B$ be a cofibration. Then*

$$T^n(p) = \{(b_0, b_1, ..., b_n) \in B^{n+1} \,|\, b_i \in E, \text{ for some } i\}$$

and the natural inclusion $\kappa_n : T^n(p) \rightarrowtail B^{n+1}$ is a cofibration.

Rigorously, this is an equality up to homotopy, since $T^n(p)$ is only defined up to homotopy. However, in the sequel, when p is cofibration, the notation $T^n(p)$ will denote exactly this subspace of B^{n+1}. Notice that if $E = *$, then $T^n(p)$ is the usual n-fatwedge of B, given by $T^n(B) = \{(b_0, b_1, ..., b_n) \in B^{n+1} \,|\, b_i = *, \text{ for some } i\}$. Also observe that if $p : E \to B$ is any map, then we can factorize p as a cofibration $\hat{p} : E \rightarrowtail \widehat{B}$ followed by a homotopy equivalence $h : \widehat{B} \xrightarrow{\simeq} B$. In this case, we have $\mathrm{secat}(p) = \mathrm{secat}(\hat{p})$ and $T^n(p)$, $T^n(\hat{p})$ are related by a homotopy commutative diagram

$$\begin{array}{ccc} T^n(\hat{p}) & \xrightarrow[\simeq]{} & T^n(p) \\ \downarrow & & \downarrow \\ (\widehat{B})^{n+1} & \xrightarrow[h^{n+1}]{\simeq} & B^{n+1} \end{array}$$

where the horizontal arrows are homotopy equivalences. This last observation together with Corollaries 9 and 11 shows that our characterization of sectional category given in Corollary 9 is equivalent to the one given by Fassò in [9].

We end this section with a useful property of the sectional fatwedge of a cofibration.

Proposition 12. *Given any pushout of the form*

$$\begin{array}{ccc} E & \longrightarrow & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \longrightarrow & B' \end{array}$$

then, for any $n \geq 0$, there exists a pushout

$$\begin{array}{ccc} T^n(p) & \longrightarrow & T^n(p') \\ {\scriptstyle \kappa_n}\downarrow & & \downarrow{\scriptstyle \kappa'_n} \\ B^{n+1} & \longrightarrow & (B')^{n+1} \end{array}$$

We remark that this result can also be stated in terms of general homotopy pushouts, in which p and p' are not necessarily cofibrations. In the form above, it follows directly from an inductive argument and the lemma below which is certainly well-known so that we just give the idea of its proof.

Lemma 13. *Consider the following honest pushout diagrams, in which f and f' are cofibrations:*

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ X & \xrightarrow[\bar{g}]{} & Z \end{array} \qquad\qquad \begin{array}{ccc} A' & \xrightarrow{g'} & Y' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \bar{f}'} \\ X' & \xrightarrow[\bar{g}']{} & Z'. \end{array}$$

Then the cofibrations $(A\times X')\cup_{A\times A'}(X\times A')\to X\times X'$ and $(Y\times Z')\cup_{Y\times Y'}(Z\times Y')\to Z\times Z'$, induced respectively by $f\times id$ and $id\times f'$ and by $\bar{f}\times id$ and $id\times\bar{f}'$, fit in the following commutative square (with the obvious maps). Furthermore, this square is a pushout:

$$\begin{array}{ccc} (A\times X')\cup_{A\times A'}(X\times A') & \longrightarrow & (Y\times Z')\cup_{Y\times Y'}(Z\times Y') \\ \downarrow & & \downarrow \\ X\times X' & \longrightarrow & Z\times Z'. \end{array}$$

Proof. The fact that the diagram is a pushout follows from the unicity of the colimit of the following commutative diagram:

$$\begin{array}{ccccc} Y\times Z' & \longleftarrow & A\times Y' & \rightarrowtail & X\times Y' \\ \uparrow & & \uparrow & & \uparrow \\ A\times X' & \longleftarrow\!\!\prec & A\times A' & \rightarrowtail & X\times A' \\ \| & & \downarrow & & \downarrow \\ A\times X' & = & A\times X' & \rightarrowtail & X\times X'. \end{array}$$

□

2 Weak sectional category.

For any $n\geq 0$ we shall denote by $B^{[n+1]}$ the $(n+1)$-fold smash-product and by $\bar{\Delta}_{n+1}=\bar{\Delta}^B_{n+1}:B\to B^{n+1}\to B^{[n+1]}$ the reduced diagonal. Recall that the weak category of B, $\mathrm{wcat}(B)$, introduced by Berstein and Hilton [2], is the least integer n such that $\bar{\Delta}_{n+1}$ is homotopically trivial. This is a lower bound for the L-S category of B since $\mathrm{cat}(B)\leq n$ if and only if the diagonal $\Delta^B_{n+1}:B\to B^{n+1}$ lifts, up to homotopy, in the fatwedge $T^n(B)=\{(b_0,b_1,...,b_n)\in B^{n+1}: b_i=*,\text{ for some } i\}$ and $B^{[n+1]}=B^{n+1}/T^n(B)$.

Following the same idea, we introduce here the weak sectional category of a map p, which will be a lower bound for the sectional category and a generalization of the weak category of Berstein and Hilton. This section is then devoted to the study of this invariant. In particular, as $\mathrm{wcat}(B)$ is, in general, a better lower bound for $\mathrm{cat}(B)$ than the cuplength of B (recall that, for any commutative ring π, $\mathrm{cuplength}(B)$ is the nilpotency of the reduced cohomology $\tilde{H}^*(B;\pi)$, that is the least integer n such that all $(n+1)$-fold cup products vanish in $\tilde{H}^*(B;\pi)$), we will show that the weak sectional category of p is a better lower bound for $\mathrm{secat}(p)$ than the classical cohomological lower bound nil ker p^* given by the nilpotency of the ideal $\ker p^* \subset \tilde{H}^*(B;\pi)$.

2.1 Definition and a characterization

Definition 14. *Let $p : E \to B$ be a map and, for any integer n, let C_{κ_n} be the homotopy cofibre of the n-sectional fat wedge of p, $\kappa_n : T^n(p) \to B^{n+1}$. If $l_n : B^{n+1} \to C_{\kappa_n}$ denotes the induced map, then the weak sectional category of p, denoted by* $\mathrm{wsecat}(p)$*, is the least integer n (or ∞) such that the composition*

$$B \xrightarrow{\Delta_{n+1}} B^{n+1} \xrightarrow{l_n} C_{\kappa_n}$$

is homotopically trivial.

It follows directly from the definition that, if B is normal, then $\mathrm{wsecat}(p) \leq \mathrm{secat}(p)$ and that, if $E = *$, then $\mathrm{wsecat}(p : * \to B) = \mathrm{wcat}(B)$. We also observe that if $p = h \circ \hat{p}$, where $\hat{p}$ is a cofibration and h is a homotopy equivalence, then $\mathrm{wsecat}(p) = \mathrm{wsecat}(\hat{p})$.

The next result is a characterization of $\mathrm{wsecat}(p)$ in terms of the homotopy cofibre of p which will be useful in order to establish the properties of the weak sectional category that will be seen in the sequel.

Proposition 15. *Let $p : E \to B$ be a map, C_p be its homotopy cofibre, and $j : B \to C_p$ be the induced map. Then* $\mathrm{wsecat}(p) \leq n$ *if and only if the map* $B \xrightarrow{\bar{\Delta}_{n+1}} B^{[n+1]} \xrightarrow{j^{[n+1]}} C_p^{[n+1]}$ *is homotopically trivial.*

Proof. Let $\widehat{B}$ be the mapping cylinder of p and $E \rightarrowtail^{\hat{p}} \widehat{B} \xrightarrow[\simeq]{h} B$ be the associated factorization of p in a cofibration followed by a homotopy equivalence. We have $\mathrm{wsecat}(p) = \mathrm{wsecat}(\hat{p})$, $C_p = \widehat{B}/E$ and the identification map $\rho : \widehat{B} \to C_p$ is naturally homotopic to jh. Applying Proposition 12 to the diagram

$$\begin{array}{ccc} E & \longrightarrow & * \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow \\ \widehat{B} & \xrightarrow{\rho} & C_p \end{array}$$

we have, for each n, the following pushout:

$$\begin{array}{ccc} T^n(\hat{p}) & \longrightarrow & T^n(C_p) \\ \downarrow & & \downarrow \\ \widehat{B}^{n+1} & \xrightarrow{\rho^{n+1}} & C_p^{n+1} \end{array}$$

The vertical maps are cofibrations and, by taking the quotients, we get the commutative diagram:

$$\begin{array}{ccc} \widehat{B}^{n+1} & \xrightarrow{\rho^{n+1}} & C_p^{n+1} \\ \downarrow{\scriptstyle q^{\hat{p}}_{n+1}} & & \downarrow{\scriptstyle q^{C_p}_{n+1}} \\ \widehat{B}^{n+1}/T^n(\hat{p}) & \xrightarrow{\simeq} & C_p^{[n+1]} \end{array}$$

Therefore, $\mathrm{wsecat}\,(\hat{p}) \leq n$ if and only if $q^{C_p}_{n+1}\rho^{n+1}\Delta^{\widehat{B}}_{n+1} \simeq *$ and, since $q^{C_p}_{n+1}\rho^{n+1} = \rho^{[n+1]}q^{\widehat{B}}_{n+1}$, we have $\mathrm{wsecat}\,(\hat{p}) \leq n$ if and only if $\rho^{[n+1]}\bar{\Delta}^{\widehat{B}}_{n+1} \simeq *$. Now, using the naturality of the reduced diagonal and the fact that $\rho \simeq jh$ we obtain $\rho^{[n+1]}\bar{\Delta}^{\widehat{B}}_{n+1} \simeq j^{[n+1]}h^{[n+1]}\bar{\Delta}^{\widehat{B}}_{n+1} \simeq j^{[n+1]}\bar{\Delta}^{B}_{n+1}h$. Since h is a homotopy equivalence, we finally have $\mathrm{wsecat}\,(\hat{p}) \leq n$ if and only if $j^{[n+1]}\bar{\Delta}^{B}_{n+1} \simeq *$. □

As a direct consequence of this characterization we get

Corollary 16. *For any map $p : E \to B$ one has* $\mathrm{wsecat}\,(p) \leq \mathrm{wcat}\,(B)$.

2.2 Weak sectional category and homotopy pushouts

Now we establish an important relation between the weak sectional category and homotopy pushouts. This relation will have interesting consequences.

Theorem 17. Consider any homotopy commutative diagram

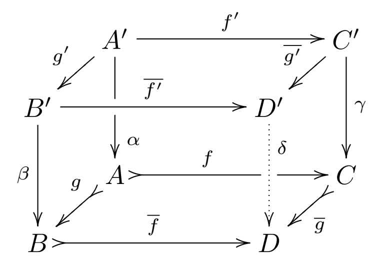
$$\begin{array}{ccccc} B' & \xleftarrow{g'} & A' & \xrightarrow{f'} & C' \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ B & \xleftarrow[g]{} & A & \xrightarrow[f]{} & C \end{array}$$

If $\delta : D' \to D$ is the induced map between the homotopy pushouts of the corresponding rows of the diagram, then

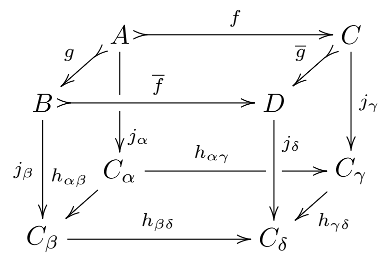
$$\mathrm{wsecat}(\delta) \leq \mathrm{wsecat}(\beta) + \mathrm{wsecat}(\gamma) + 1.$$

In particular, if $A' = B' = C' = *$, we have $\mathrm{wcat}(D) \leq \mathrm{wcat}(B) + \mathrm{wcat}(C) + 1$.

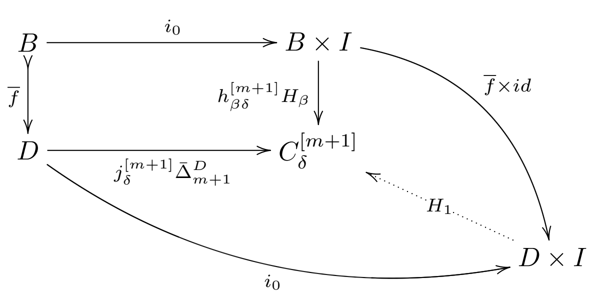
Proof. Using factorizations through a cofibration followed by a homotopy equivalence, we can suppose without losing generality, that g and f are cofibrations and that D is the honest pushout of f and g. We obtain the following homotopy commutative cube, where $\overline{f}$ and $\overline{g}$ (the cobase change of f and g, respectively) are also cofibrations:



On the other hand, taking homotopy cofibres of the respective columns, it is easy to obtain a cube in which all the vertical faces (by assumption also the top face) are strictly commutative



Now suppose that $\text{wsecat}(\beta) \leq m$, $\text{wsecat}(\gamma) \leq n$ and take nullhomotopies $H_\beta : j_\beta^{[m+1]}\bar{\Delta}^B_{m+1} \simeq *$, $H_\gamma : j_\gamma^{[n+1]}\bar{\Delta}^C_{n+1} \simeq *$. From the homotopy extension property of the cofibration $\overline{f} : B \rightarrowtail D$ we obtain a commutative diagram



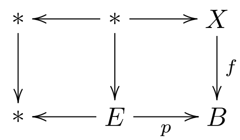
Similarly, we also obtain a homotopy $H_2 : D \times I \to C_\delta^{[n+1]}$ such that $H_2 i_0 = j_\delta^{[n+1]}\bar{\Delta}^D_{n+1}$ and $H_2(\overline{g} \times id) = h_{\gamma\delta}^{[n+1]}H_\gamma$. Then it is easy to check that the composite $D \times I \overset{(H_1,H_2)}{\longrightarrow} C_\delta^{[m+1]} \times C_\delta^{[n+1]} \longrightarrow C_\delta^{[m+n+2]}$ defines a homotopy between $j_\delta^{[m+n+2]} \bar{\Delta}^D_{m+n+2}$ and the trivial map.

□

Corollary 18. Consider $E \xrightarrow{p} B \xrightarrow{j} C_p$ a cofibre sequence. If $f : X \to B$ is any map then

$$\text{wsecat}(jf) \leq \text{wsecat}(f) + 1$$

Proof. Apply the above result to the following commutative diagram

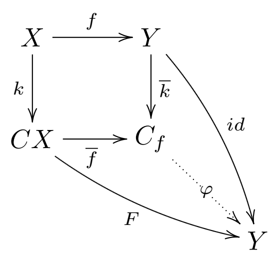


Observe that in this case, the induced map between the homotopy pushouts verifies $\delta \simeq jf$. □

Lemma 19. Let $f : X \to Y$ be any map such that $f \simeq *$. Then

$$\text{wsecat}(f) = \text{wcat}(Y).$$

Proof. By Corollary 16, it suffices to prove that $\text{wsecat}(f) \geq \text{wcat}(Y)$. Since $f \simeq *$ there is a factorization $f = Fk$, where $F : CX \to Y$ and $k : X \to CX$ is the natural inclusion on the cone of X. Taking the following pushout,



we get a map $\varphi : C_f \to Y$ which is a retraction of the inclusion $\overline{k} : Y \to C_f$. We thus have $\varphi^{[n+1]}(\overline{k})^{[n+1]} = id$. Therefore, if the map $Y \xrightarrow{\bar{\Delta}^Y_{n+1}} Y^{[n+1]} \xrightarrow{\overline{k}^{[n+1]}} C_f^{[n+1]}$ is homotopically trivial, then so is the reduced diagonal $\bar{\Delta}^Y_{n+1} : Y \to Y^{[n+1]}$. By Proposition 15, we obtain the expected inequality. □

Corollary 20. Consider $E \xrightarrow{p} B \xrightarrow{j} C_p$ a cofibre sequence. Then

$$\text{wcat}(C_p) \leq \text{wsecat}(p) + 1$$

Proof. Apply Corollary 18 and Lemma 19 when $f = p$.. □

Observe that this inequality is an improvement of the classical inequality $\text{wcat}(C_p) \leq \text{wcat}(B) + 1$.

2.3 Some remarkable inequalities

In the two previous sections we have seen that, for a map $p : E \to B$ with homotopy cofibre C_p, we have $\mathrm{wsecat}\,(p) \leq \mathrm{wcat}\,(B)$ and $\mathrm{wsecat}\,(p) \geq \mathrm{wcat}\,(C_p) - 1$. Actually these two inequalities fit in a set of inequalities we collect in the following statement:

Theorem 21. *If $p : E \to B$ is a map and C_p its homotopy cofibre, then*

(a) $\mathrm{wsecat}\,(p) \leq \mathrm{wcat}\,(B)$

(b) $\mathrm{wsecat}\,(p) \leq \mathrm{wcat}\,(C_p)$

(c) $\mathrm{wsecat}\,(p) \geq \mathrm{wcat}\,(C_p) - 1$

(d) $\mathrm{wsecat}\,(p) \geq \mathrm{nil}\,\ker p^*$ *(for any commutative ring π)*

(e) If the map $p : E \to B$ admits a homotopy retraction, then

$$\mathrm{wsecat}\,(p) = \mathrm{wcat}\,(C_p) \;\; \textit{and} \;\; \mathrm{nil}\,\ker p^* = \mathrm{cuplength}\,(C_p).$$

Observe that the inequalities (a), (b), (c) and (d) can be summarized in the following way:

$$\max(\mathrm{wcat}\,(C_p) - 1, \mathrm{nil}\,\ker p^*) \leq \mathrm{wsecat}\,(p) \leq \min(\mathrm{wcat}\,(C_p), \mathrm{wcat}\,(B))$$

After the proof of the theorem, we will see, through examples, that all the inequalities (a), (b), (c) and (d) can be strict. In particular, this shows that $\mathrm{wsecat}\,(p)$ is a better lower bound for $\mathrm{secat}\,(p)$ than the classical cohomological lower bound $\mathrm{nil}\,\ker p^*$.

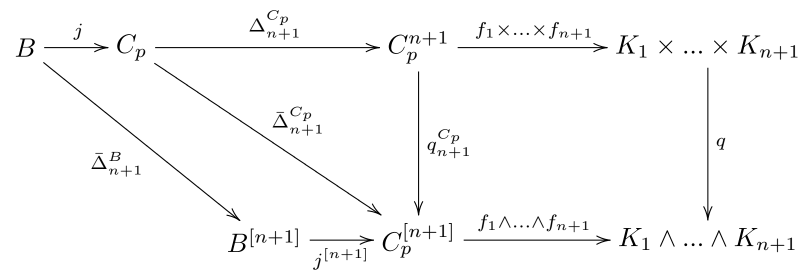
Proof. We just have to prove (b), (d) and (e). We use the characterization of wsecat established in Proposition 15. We first prove (b). By naturality of the diagonal the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\bar{\Delta}^{B}_{n+1}} & B^{[n+1]} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j^{[n+1]}} \\ C_p & \xrightarrow{\bar{\Delta}^{C_p}_{n+1}} & C_p^{[n+1]} \end{array}$$

Therefore if $\mathrm{wcat}\,(C_p) \leq n$, then we have $\mathrm{wsecat}\,(p) \leq n$.
We now prove (d). Assume that $\mathrm{wsecat}(p) \leq n$ and take $x_1, ..., x_{n+1} \in \ker(p^*) \subset \tilde{H}^*(B)$ where $\tilde{H}^*$ stands for the reduced cohomology with coefficients in a commutative ring π. From the cofibre sequence $E \xrightarrow{p} B \xrightarrow{j} C_p$ we have an exact sequence in reduced cohomology

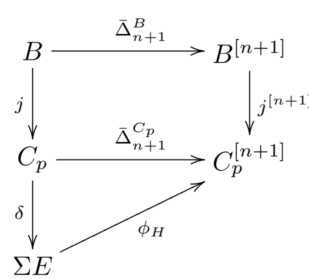
$$\tilde{H}^*(C_p) \xrightarrow{j^*} \tilde{H}^*(B) \xrightarrow{p^*} \tilde{H}^*(E)$$

so we can consider $y_1, ..., y_{n+1} \in \tilde{H}^*(C_p)$ such that $j^*(y_i) = x_i$, $i = 1, ..., n+1$. Now take, for each i, a representative $f_i : C_p \to K_i$ of y_i (where $K_i = K(\pi, m_i)$ is the Eilenberg-MacLane space of type (π, m_i), for certain m_i) and the following commutative diagram:



Then $x_1 \cup ... \cup x_{n+1} = j^*(y_1 \cup ... \cup y_{n+1})$ has as a representative the composition of $q(f_1 \times ... \times f_{n+1})\Delta^{C_p}_{n+1} j$ with certain map $K_1 \wedge ... \wedge K_{n+1} \to K(\pi, m_1 + ... + m_{n+1})$ (see, for instance, [1]). But the first map is already homotopy trivial since $j^{[n+1]}\bar{\Delta}^B_{n+1} \simeq *$. This completes the proof of (d). We also observe that if $y_1 \cup ... \cup y_{n+1} = 0$, then $x_1 \cup ... \cup x_{n+1} = j^*(y_1 \cup ... \cup y_{n+1}) = 0$. In other words, nil ker $p^* \leq$ cuplength(C_p).

Finally we prove (e). Suppose that wsecat$(p) \leq n$ and let H be a homotopy between $j^{[n+1]}\bar{\Delta}^B_{n+1} = \bar{\Delta}^{C_p}_{n+1} j$ and the trivial map. This homotopy together with the choice of a homotopy equivalence between C_j and ΣE determines a map $\phi_H : \Sigma E \to C_p^{[n+1]}$ such that the lower triangle in the following diagram is homotopy commutative:



Here δ is the identification map $C_p \to C_p/B = \Sigma E$. It follows directly from this diagram that if $\delta \simeq *$ or $\phi_H \simeq *$, then wcat$(C_p) \leq n$ and we can conclude that wsecat$(p) =$ wcat(C_p). In particular, if p has a homotopy retraction r, then Σr is a homotopy retraction of Σp and by considering the Barratt-Puppe sequence

$$E \xrightarrow{p} B \xrightarrow{j} C_p \xrightarrow{\delta} \Sigma E \xrightarrow{\Sigma p} \Sigma B \longrightarrow \cdots$$

we see that $\delta \simeq \Sigma r \Sigma p \, \delta \simeq *$. This proves the first part of the statement. For the second part, we observe that if p has a homotopy retraction, then p^* is surjective and $\ker p^* = \tilde{H}^*(C_p)$. Therefore $\mathrm{nil}\ker p^* = \mathrm{nil}\,\tilde{H}^*(C_p) = \mathrm{cuplength}\,(C_p)$.

□

The following examples show that the inequalities (a), (b), (c) and (d), in this order, can be strict.

Examples

1. If B is a space with $\mathrm{wcat}\,(B) > 0$ then the weak sectional category of the projection $p : A \times B \to B$, where A is any space, satisfies $\mathrm{wsecat}\,(p) = 0 < \mathrm{wcat}\,(B)$.
2. Let p be the Hopf map $S^3 \to S^2$. We have $\mathrm{wsecat}\,(p) \leq \mathrm{wcat}\,(S^2) = 1$ but $\mathrm{wcat}\,(C_p) = \mathrm{wcat}\,(\mathbb{CP}^2) = 2$.
3. If p admits a homotopy retraction, then $\mathrm{wsecat}\,(p) = \mathrm{wcat}\,(C_p) > \mathrm{wcat}\,(C_p) - 1$. A concrete example is given by taking for p an inclusion $B \to B \vee A$.
4. Recall that the symplectic group $\mathrm{Sp}(2)$ admits a cellular decomposition of the form $S^3 \cup_\alpha e^7 \cup_\beta e^{10}$. It has been proved by Berstein and Hilton [2] that $\mathrm{wcat}\,(S^3 \cup_\alpha e^7) = \mathrm{cat}\,(S^3 \cup_\alpha e^7) = 2$ and by Schweitzer [14] that $\mathrm{wcat}\,(\mathrm{Sp}(2)) = \mathrm{cat}\,(\mathrm{Sp}(2)) = 3$. Take $p = \beta : S^9 \to S^3 \cup_\alpha e^7$. Since $\mathrm{wcat}\,(S^3 \cup_\alpha e^7) = 2$, we have $\mathrm{wsecat}\,(p) \leq 2$. On the other hand, since $C_p = \mathrm{Sp}(2)$, we have $\mathrm{wsecat}\,(p) \geq \mathrm{wcat}\,(C_p) - 1 = 2$. Therefore $\mathrm{wsecat}\,(p) = 2$. But $\ker(p^*) = \tilde{H}^*(S^3 \cup_\alpha e^7)$ and $\mathrm{nil}\,(\ker p^*) = 1$.

Our last result for this section establishes the equality between the weak sectional category and the sectional category under some restrictions on the dimension and connectivity. This kind of results is classical in the theory of Lusternik-Schnirelmann category (see, for instance, [4] [Section 2.8]) and is based on Blakers-Massey Theorem that, in this context, it is appropriate to state in the following form:

Lemma 22 (Blakers-Massey Theorem). *Let $A \xrightarrow{j} B \xrightarrow{q} C$ be a cofibre sequence where all spaces are simply connected, A is $(a-1)$-connected and C is $(c-1)$-connected. Let F be the homotopy fibre of q, $\iota : F \to B$ the induced map and $d : A \to F$ a lifting of j (i.e. $id \simeq j$). Then d is an $(a + c - 2)$-equivalence.*

Theorem 23. Let $p : E \to B$ be any map, where E and B are (q-1)-connected CW-complexes ($q \geq 2$). If dim(B)=N and either one of the following conditions is satisfied

(i) $N \leq q(\mathrm{wsecat}\,(p) + 2) - 2$

(ii) $N \leq q(\mathrm{secat}\,(p) + 1) - 2$,

then $\mathrm{secat}\,(p) = \mathrm{wsecat}\,(p)$.

Proof. For the proof of (i) suppose that $\mathrm{wsecat}(p) = n$ and consider the following diagram

$$\begin{array}{ccccc} \widetilde{F}_{n+1} & \xrightarrow{\iota_n} & B^{n+1} & \xrightarrow{l_n} & C_{\kappa_n} \\ \uparrow{\scriptstyle \widetilde{\kappa}_n} & \nearrow{\scriptstyle \kappa_n} & & & \\ T^n(p) & & & & \end{array}$$

where $\widetilde{F}_{n+1}$ the homotopy fiber of l_n and $\widetilde{\kappa}_n$ is the map induced by the homotopy fiber property. By the same property we can also consider a lift $\widetilde{\Delta}^B_{n+1} : B \to \widetilde{F}_{n+1}$ of the diagonal map Δ^B_{n+1}. Taking into account that $T^n(p)$ is (q-1)-connected and C_{κ_n} is (q(n+1)-1)-connected then, by the Blakers-Massey Theorem, $\widetilde{\kappa}_n$ is a (q(n+2)-2)-equivalence. So, by the hypothesis on the dimension of B, we have that

$$(\widetilde{\kappa}_n)_* : [B, T^n(p)] \to [B, \widetilde{F}_{n+1}]$$

is a surjection and therefore there exists a map $\widehat{\Delta}^B_{n+1} : B \to T^n(p)$ such that $\widetilde{\kappa}_n \widehat{\Delta}^B_{n+1} \simeq \widetilde{\Delta}^B_{n+1}$. Composing ι_n to both sides we obtain a factorization $\kappa_n \widehat{\Delta}^B_{n+1} \simeq \Delta^B_{n+1}$; that is, $\mathrm{secat}(p) \leq n$.

For (ii) suppose that $\mathrm{secat}(p) = n$ and that $\mathrm{wsecat}(p) \leq n-1$. By a similar argument to that above one can find a factorization $\kappa_{n-1}\widehat{\Delta}^B_n \simeq \Delta^B_n$, which is a contradiction. □

3 Weak topological complexity.

Studying the motion planning problem by using topological techniques, M. Farber introduced in [7] and [8] the topological complexity of any space X, $\mathrm{TC}(X)$, as the sectional category of the evaluation fibration $\pi_X : X^I \to X \times X$, $\pi_X(\alpha) = (\alpha(0), \alpha(1))$. Broadly speaking, this invariant measures the discontinuity of any motion planner in the space. As the topological complexity of a space X is the sectional category of the fibration π_X we may consider our weak version of sectional category to obtain a *new* lower bound of this homotopy invariant. Namely, we define the *weak topological complexity* of X as

$$\mathrm{wTC}(X) := \mathrm{wsecat}(\pi_X)$$

Observe that, since π_X is the mapping path fibration associated to the diagonal map $\Delta_X : X \to X \times X$, we have that $\mathrm{wTC}(X) = \mathrm{wsecat}(\Delta_X)$. We also note that in many cases the n-fatwedge of Δ_X has the nice description given in Section 2. Namely, for locally equiconnected spaces (those spaces in which Δ_X is a cofibration) $T^n(\Delta_X)$ has, up to homotopy equivalence, the following explicit expression

$$T^n(\Delta_X) = \{(y_0, ..., y_n) \in (X \times X)^{n+1} \mid y_i \in \Delta_X(X) \text{ for some } i\}$$

Under the same condition on X, the homotopy cofiber of Δ_X, C_{Δ_X}, is homotopically equivalent to the quotient space

$$(X \times X)/\Delta_X(X).$$

Note that the class of locally equiconnected spaces is not very restrictive. For instance, the CW-complexes and the metrizable topological manifolds fit on such class of spaces.

The next two results are just a specialization of Theorem 21 and Theorem 23, respectively, and therefore their proofs are omitted. Recall that, if $\tilde{H}^*$ stands for the reduced cohomology with coefficients in a field $\mathbf{K}$, then M. Farber [7] proved that $\text{nil}\ker(\Delta_X)^* = \text{nil}\ker \cup$, where $\cup : \tilde{H}^*(X) \otimes \tilde{H}^*(X) \to \tilde{H}^*(X)$ denotes the usual cup-product.

Theorem 24. *Let X be any space. Then*

(a) $\text{wTC}(X) \leq \text{wcat}(X \times X)$

(b) $\text{wTC}(X) \geq \text{nil}\ker \cup$

(c) $\text{wTC}(X) = \text{wcat}(C_{\Delta_X})$ *(and* $\text{nil}\ker \cup = \text{cuplength}(C_{\Delta_X})$*).*

Theorem 25. Let X be any (q-1)-connected CW-complex ($q \geq 2$). If dim(X)=N and either one of the following conditions is satisfied

(i) $N \leq \frac{q(\text{wTC}(X)+2)}{2} - 1$

(ii) $N \leq \frac{q(\text{TC}(X)+1)}{2} - 1$,

then $\text{TC}(X) = \text{wTC}(X)$.

Finally we give two concrete computations. The first one (Proposition 26 below) consists of the explicit determination of the homotopy cofibre of the diagonal map $\Delta_{S^n} : S^n \to S^n \times S^n$ of a sphere. Using the results of [2] together with the classical results on the Hopf invariant of the Whitehead product $[\iota_n, \iota_n]$ (where ι_n denotes the homotopy class of the identity of S^n) it is then possible to deduce that $\text{wTC}(S^n)(= \text{wcat}(C_{\Delta_{S^n}}))$ is 1 if n is odd and is 2 if n is even. The previous theorem permits to recover the result by Farber that the topological complexity of an odd dimensional sphere is 1 while that of an even dimensional sphere is 2.

Proposition 26. *The homotopy cofibre of $\Delta_{S^n} : S^n \to S^n \times S^n$ is homotopy equivalent to $S^n \cup_{[\iota_n, \iota_n]} e^{2n}$.*

Proof. Let denote by $\nu : S^n \to S^n \vee S^n$ the pinch map, by $j : S^n \vee S^n \to S^n \times S^n$ the inclusion and by $\tilde{\nabla}$ the map $S^n \vee S^n \xrightarrow{(id,-id)} S^n$. It is sufficient to establish the result for the homotopy cofibre of $j\nu$ since this composite is homotopic to the diagonal map Δ_{S^n}. It is not hard to check that the following homotopy commutative diagram is a homotopy pushout

$$\begin{array}{ccc} S^n & \xrightarrow{\nu} & S^n \vee S^n \\ \downarrow & & \downarrow{\scriptstyle \tilde{\nabla}} \\ * & \longrightarrow & S^n \end{array}$$

We can thus consider the following diagram in which w is the attaching map of the top cell of $S^n \times S^n$ and the two lower squares are pushouts and homotopy pushouts

$$\begin{array}{ccccc} & & S^n & \longrightarrow & * \\ & & \downarrow{\scriptstyle \nu} & & \downarrow \\ S^{2n-1} & \xrightarrow{w} & S^n \vee S^n & \xrightarrow{\tilde{\nabla}} & S^n \\ \downarrow & & {\scriptstyle j}\downarrow & & \downarrow \\ CS^{2n-1} & \longrightarrow & S^n \times S^n & \longrightarrow & Z \end{array}$$

By composition we see that, on the one hand, Z is homotopy equivalent to the homotopy cofibre of $j\nu$ and, on the other hand, Z is the homotopy cofibre of $\tilde{\nabla}w$ whose homotopy class is the Whitehead product $[\iota_n, -\iota_n] = -[\iota_n, \iota_n]$. We then have $Z \simeq S^n \cup_{-[\iota_n,\iota_n]} e^{2n} \simeq S^n \cup_{[\iota_n,\iota_n]} e^{2n}$, where the last equivalence is induced by $-id : S^{2n-1} \to S^{2n-1}$. $\square$

Our second concrete computation has the objective to show, through an example, that wTC is, in general, a better lower bound for the topological complexity than nil ker $\cup$. Before giving this example, we first prove a useful general result about weak category.

Proposition 27. *Let $f : A \to B$ be a map between $(q-1)$-connected CW-complexes ($q \geq 1$). If f is an r-equivalence,* $\mathrm{wcat}(A) \geq k$ *and* $\dim(A) \leq r + q(k-1) - 1$ *then* $\mathrm{wcat}(B) \geq \mathrm{wcat}(A)$.

Proof. Since $\mathrm{wcat}(A) \geq k$ the k-reduced diagonal $\bar{\Delta}_k^A : A \to A^{[k]}$ is not homotopically trivial. By the naturality of this diagonal we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\bar{\Delta}_k^A} & A^{[k]} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{[k]}} \\ B & \xrightarrow[\bar{\Delta}_k^B]{} & B^{[k]} \end{array}$$

Since f is an r-equivalence and A and B are $(q-1)$-connected CW-complexes we obtain that $f^{[k]}$ is an $r + q(k-1)$ equivalence. The assumption on the dimension of A permits thus to assert that

$$(f^{[k]})_* : [A, A^{[k]}] \to [A, B^{[k]}]$$

is injective. Since $\bar{\Delta}_k^A$ is not homotopically trivial we conclude that $\bar{\Delta}_k^B \circ f = f^{[k]} \circ \bar{\Delta}_k^A$ and $\bar{\Delta}_k^B$ are not homotopically trivial. Therefore $\mathrm{wcat}(B) \geq k$. □

Proposition 28. *Let $X := S^3 \cup_\alpha e^7$ where $\alpha : S^6 \to S^3$ is the Blakers-Massey element (that is, X is the 7-skeleton of Sp(2)). For this space, we have* nil ker $\cup = 2$ *and* $\mathrm{wTC}(X) \geq 3$.

Proof. Since $H^*(X) = H^*(S^3 \vee S^7)$, it is easy to check that nil ker $\cup = 2$. In order to prove that $\mathrm{wTC}(X) = \mathrm{wcat}(C_{\Delta_X}) \geq 3$ we proceed in three steps. Let C be the homotopy cofibre of the composition

$$(incl \times id)\Delta_{S^3} : S^3 \xrightarrow{\Delta_{S^3}} S^3 \times S^3 \xrightarrow{incl \times id} X \times S^3 .$$

The first step aims to see that it suffices to show that $\mathrm{wcat}(C) \geq 3$. The second step shows that C fits in a special diagram and the third step gives the proof that $\mathrm{wcat}(C) \geq 3$.

Step 1. Let $f : C \to C_{\Delta_X}$ be the map induced by the horizontal maps of the following square when we take the homotopy cofibres of the vertical maps.

$$\begin{array}{ccc} S^3 & \xrightarrow{incl} & X \\ {\scriptstyle (incl\times id)\Delta_{S^3}}\downarrow & & \downarrow{\scriptstyle \Delta_X} \\ X \times S^3 & \xrightarrow[id \times incl]{} & X \times X \\ \downarrow & & \downarrow \\ C & \overset{}{\underset{f}{\dashrightarrow}} & C_{\Delta_X} \end{array}$$

Taking into account that the map $X \times X \to C_{\Delta_X}$ induces a surjective morphism in homology (because Δ_X admits a retraction), it is easy to see that $H_i(f)$ is an isomorphism for $i \leq 5$ and is surjective for $i = 6$. Since C and C_{Δ_X} are 2-connected (and thus 1-connected) we can conclude that f is a 6-equivalence. Since $\dim(C) = 10$, we apply Proposition 27 with $q = 3$ and $r = 6$ to obtain that if $\mathrm{wcat}(C) \geq 3$, then $\mathrm{wcat}(C_{\Delta_X}) \geq 3$.

Step 2. Consider now the map $S^3 \times S^3 \xrightarrow{\mu'} S^3$ given by $\mu'(x, y) = x \cdot y^{-1}$. Since $\mu' \Delta_{S^3}$ is trivial, μ' factors through $C_{\Delta_{S^3}}$ and we have the following commutative diagram

$$\begin{array}{ccc} S^3 \times S^3 & \xrightarrow{\varphi} & C_{\Delta_{S^3}} \\ & {\scriptstyle \mu'}\searrow & \downarrow{\scriptstyle \rho} \\ & & S^3 \end{array}$$

Let $i_1 : S^3 \to S^3 \times S^3$ be the inclusion of the first factor. We will see that there exists a commutative diagram of the form

$$\begin{array}{ccc} C_{\Delta_{S^3}} \cup_{\varphi i_1 \alpha} e^7 & \rightarrowtail & C \\ \downarrow & & \downarrow \\ X & \rightarrowtail & \mathrm{Sp}(2) \end{array}$$

which is a pushout and in which the map $X \to \mathrm{Sp}(2)$ is the inclusion and the map $C \to \mathrm{Sp}(2)$ is a 5-equivalence. We first prove that the pushout of the cofibration $S^3 \times S^3 \xrightarrow{incl \times id} X \times S^3$ and the map $S^3 \times S^3 \xrightarrow{\mu'} S^3$ is the group $\mathrm{Sp}(2)$. Since $\mathrm{Sp}(2)$ is the total space of the S^3-principal bundle over S^7 classified by the adjoint of $\alpha : S^6 \to S^3 = \Omega B S^3$, we have the following pushout diagram

$$\begin{array}{ccc} S^6 \times S^3 & \xrightarrow{incl \times id} & CS^6 \times S^3 \\ \chi \downarrow & & \downarrow \\ S^3 & \rightarrowtail & \mathrm{Sp}(2) \end{array}$$

where χ is the composition $S^6 \times S^3 \xrightarrow{\alpha \times id} S^3 \times S^3 \xrightarrow{mult} S^3$ and the map $S^3 \to \mathrm{Sp}(2)$ is the inclusion. This diagram admits the following decomposition in which the two upper squares are pushouts:

$$\begin{array}{ccc} S^6 \times S^3 & \xrightarrow{incl \times id} & CS^6 \times S^3 \\ {\scriptstyle \alpha \times id} \downarrow & & \downarrow \\ S^3 \times S^3 & \xrightarrow{incl \times id} & X \times S^3 \\ {\scriptstyle id \times inv} \downarrow & & \downarrow {\scriptstyle id \times inv} \\ S^3 \times S^3 & \xrightarrow{incl \times id} & X \times S^3 \\ {\scriptstyle \mu'} \downarrow & & \downarrow \\ S^3 & \rightarrowtail & \mathrm{Sp}(2) \end{array}$$

We then conclude that the bottom square is also a pushout diagram. We now decompose μ' in $\rho\varphi$ and consider the pushout of φ and the cofibration $incl \times id : S^3 \times S^3 \to X \times S^3$. This pushout is exactly the homotopy cofibre of $(incl \times id)\Delta_{S^3}$, that is C. In this way the bottom square of the previous diagram admits the following decomposition into two pushouts:

$$\begin{array}{ccc}
S^3 \times S^3 & \xrightarrow{incl \times id} & X \times S^3 \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \hat{\varphi}} \\
C_{\Delta_{S^3}} & \longrightarrow & C \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \hat{\rho}} \\
S^3 & \longrightarrow & \mathrm{Sp}(2)
\end{array}$$

Now we decompose the cofibration $incl \times id : S^3 \times S^3 \to X \times S^3$ in the two following cofibrations

$$S^3 \times S^3 \rightarrowtail S^3 \times S^3 \bigcup_{S^3 \vee S^3} X \vee S^3 \rightarrowtail X \times S^3$$

and observe that the indermediate space is exactly the homotopy cofibre of $i_1\alpha : S^6 \to S^3 \times S^3$. By constructing the pushout of the cofibration $S^3 \times S^3 \to S^3 \times S^3 \bigcup_{S^3 \vee S^3} X \vee S^3$ with first φ and secondly $\rho\varphi$ we then obtain the homotopy cofibres of $\varphi i_1 \alpha$ and $\rho\varphi i_1 \alpha = \alpha$ (note that $\rho\varphi i_1 = \mu' i_1 = id$) and the following commutative diagram in which each square is a pushout.

$$\begin{array}{ccccc}
S^3 \times S^3 & \rightarrowtail & S^3 \times S^3 \bigcup_{S^3 \vee S^3} X \vee S^3 & \rightarrowtail & X \times S^3 \\
\downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow \\
C_{\Delta_{S^3}} & \rightarrowtail & C_{\Delta_{S^3}} \cup_{\varphi i_1 \alpha} e^7 & \rightarrowtail & C \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \tilde{\rho}} & & \downarrow{\scriptstyle \hat{\rho}} \\
S^3 & \rightarrowtail & X & \rightarrowtail & \mathrm{Sp}(2)
\end{array}$$

The right bottom pushout is the expected one. Observe that the inclusion $S^3 \times S^3 \bigcup_{S^3 \vee S^3} X \vee S^3 \hookrightarrow X \times S^3$ is a 9-equivalence. Thus so is the map $X \to \mathrm{Sp}(2)$. In other words this map is the inclusion. On the other hand the map $\hat{\rho}$ has the same connectivity as ρ. In order to justify that ρ is a 5-equivalence we just recall that $C_{\Delta_{S^3}} = S^3 \cup_{[\iota_3, -\iota_3]} e^6 \simeq S^3 \vee S^6$ because the Whitehead products are trivial in S^3 and that ρ satisfies $\rho\varphi i_1 = id$.

Step 3. We finally prove that $\mathrm{wcat}(C) \geq 3$. The homotopy cofibre of the inclusion $X \to \mathrm{Sp}(2)$ is homotopically equivalent to S^{10}. Therefore, from the pushout of the previous step, we have the following homotopy commutative diagram.

$$\begin{array}{ccc}
C & \xrightarrow{\partial} & S^{10} \\
\downarrow{\scriptstyle \hat{\rho}} & & \| \\
\mathrm{Sp}(2) & \xrightarrow[\delta]{} & S^{10}
\end{array}$$

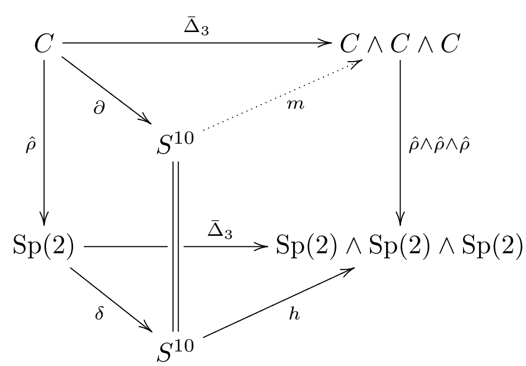
On the other hand, by naturality, the following diagram is commutative

$$\begin{array}{ccc} C & \xrightarrow{\bar{\Delta}_3} & C \wedge C \wedge C \\ {\scriptstyle \hat{\rho}}\downarrow & & \downarrow{\scriptstyle \hat{\rho}\wedge\hat{\rho}\wedge\hat{\rho}} \\ \mathrm{Sp}(2) & \xrightarrow{\bar{\Delta}_3} & \mathrm{Sp}(2) \wedge \mathrm{Sp}(2) \wedge \mathrm{Sp}(2) \end{array}$$

Since $\mathrm{wcat}\,(X) \leq 2$, the 3rd reduced diagonal of $Sp(2)$ factors through δ and we have a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}(2) & \xrightarrow{\bar{\Delta}_3} & \mathrm{Sp}(2) \wedge \mathrm{Sp}(2) \wedge \mathrm{Sp}(2) \\ & {\scriptstyle \delta}\searrow \quad \nearrow{\scriptstyle h} & \\ & S^{10} & \end{array}$$

As $\mathrm{wcat}\,(\mathrm{Sp}(2)) > 2$ the map h is not homotopically trivial (actually h is known to be the composite $S^{10} \stackrel{\Sigma^7\eta}{\to} S^9 \hookrightarrow \mathrm{Sp}(2) \wedge \mathrm{Sp}(2) \wedge \mathrm{Sp}(2)$, see [4, Rem. 6.50]). Now, taking into account that $\hat{\rho}$ is a 5-equivalence and that C and $\mathrm{Sp}(2)$ are 2-connected we can check that $\hat{\rho} \wedge \hat{\rho} \wedge \hat{\rho}$ is a 11-equivalence. Therefore h lifts to a map $m : S^{10} \to C \wedge C \wedge C$ such that $(\hat{\rho} \wedge \hat{\rho} \wedge \hat{\rho})m \simeq h$. We then have the following diagram in which each face, except the top triangle, is homotopy commutative.



Since $\dim(C) = 10$, we deduce from the fact that $\hat{\rho} \wedge \hat{\rho} \wedge \hat{\rho}$ is a 11-equivalence that the top triangle is actually homotopy commutative. Suppose now that $\mathrm{wcat}\,(C) \leq 2$. This implies that the composite $m\partial$ is homotopically trivial and, therefore, that the map m factors through the homotopy cofibre of ∂. Since this homotopy cofibre is homotopy equivalent to $\Sigma(C_{\Delta_{S^3}} \cup_{\varphi i_1 \alpha} e^7)$ we then have a homotopy commutative diagram:

$$\begin{array}{ccc} S^{10} & \xrightarrow{m} & C \wedge C \wedge C \\ & \searrow \quad \nearrow & \\ & \Sigma(C_{\Delta_{S^3}} \cup_{\varphi i_1 \alpha} e^7) & \end{array}$$

But $C \wedge C \wedge C$ is 8-connected and $\dim(\Sigma(C_{\Delta_{S^3}} \cup_{\varphi i_1 \alpha} e^7)) = 8$. Therefore m should be homotopically trivial which contradicts the fact that h is not homotopically trivial. We then conclude that $\mathrm{wcat}(C) \geq 3$. □

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